Amherst College
Department of Mathematics

Math 271 — Midterm 2 Practice Exam 2 — Spring 2022

Name:____

This is a modified version of a practice exam from Fall 2018.

Read This First!

- Please read each question carefully. Show **ALL** work clearly in the space provided.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question.
- Please cross out or fully erase any work that you do not want graded.
- The point value of each question is indicated after its statement.
- No books or other references are permitted.
- Calculators are not allowed and you must show all your work.

Grading - For Administrative Use Only

Question:	1	2	3	4	5	Total
Points:	15	15	15	15	15	75
Score:						

1. True or False: (No justification necessary.)

   (a) Every injective linear transformation $T : V \to W$ is also surjective. **T** **F** [3]

   (b) If V is a finite dimensional vector space, then there is a linear transformation $T : V \to V$ such that $[T]_\alpha^\alpha = [T]_\beta^\beta$ for all α and β bases of V. Comment (2022): this problem concerns a topic we haven't discussed much this semester, so I'd be unlikely to ask this question, but it may still be useful to try to figure it out. **T** **F** [3]

   (c) If $T : P_5(\mathbb{R}) \to \mathbb{R}^5$ is linear, then T is not surjective. **T** **F** [3]

   (d) If V is a finite-dimensional vector space and W is a subspace of V, then $\dim(V) \leq \dim(W)$. **T** **F** [3]

2. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation with matrix representation [15]

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 \\ 5 & 5 & 7 & 7 & 7 \end{bmatrix}$$

with respect to the standard bases. Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$.

3. Let [15]

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The set $\beta = \{B_1, B_2, B_3, B_4\}$ is a basis for $M_{2\times 2}(\mathbb{R})$. Let $T : M_{2\times 2}(\mathbb{R}) \to \mathrm{M}_{2\times 2}(\mathbb{R})$ be given by $T(A) = A^t - 2A$, where A^t denotes the transpose of A. Take my word for it, that T is a linear transformation.

Find the matrix $[T]_\beta^\beta$.

4. Let $S : U \to V$ and $T : V \to W$ be linear transformations. Define the *composition* transformation $TS : U \to W$ by the equation $TS(\vec{u}) = T(S(\vec{u}))$ for all $\vec{u} \in U$. Prove that TS is a linear transformation. [15]

5. Let $T : V \to W$ be an injective linear transformation. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq V$ is linearly independent, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent. [15]