This is a modified version of a practice exam from Fall 2018.

Read This First!

- Please read each question carefully. Show ALL work clearly in the space provided.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question.
- Please cross out or fully erase any work that you do not want graded.
- The point value of each question is indicated after its statement.
- No books or other references are permitted.
- Calculators are not allowed and you must show all your work.

Grading - For Administrative Use Only

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1. **True or False:** (No justification necessary.)

   (a) Every injective linear transformation \( T : V \to W \) is also surjective. \( \boxed{T \quad F \quad [3]} \)
   
   \( \text{(eg. } \mathbb{R}^2 \to \mathbb{R}^3; \ T(x,y) = (x,y,0)) \)  

   (b) If \( V \) is a finite-dimensional vector space, then there is a linear transformation \( \boxed{T \quad F \quad [3]} \)
   
   \( T : V \to V \) such that \( [T]_{\alpha}^\alpha = [T]_{\beta}^\beta \) for all \( \alpha \) and \( \beta \) bases of \( V \).  
   
   Comment (2022): this problem concerns a topic we haven’t discussed much this semester, so I’d be unlikely to ask this question, but it may still be useful to try to figure it out.
   
   \( \text{(eg. the identity transformation, or the zero transformation)} \)  

   (c) If \( T : P_5(\mathbb{R}) \to \mathbb{R}^5 \) is linear, then \( T \) is not surjective. \( \boxed{T \quad F \quad [3]} \)
   
   \( \text{(dim} \ P_5(\mathbb{R}) > \text{dim} \ \mathbb{R}^5; \ \text{surjective maps exist}) \)  

   (d) If \( V \) is a finite-dimensional vector space and \( W \) is a subspace of \( V \), then \( \text{dim}(V) \leq \text{dim}(W) \). \( \boxed{T \quad F \quad [3]} \)
   
   \( \text{(instead, dim} \ W \leq \text{dim} \ V) \)
2. Let $T : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ be the linear transformation with matrix representation

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 \\
5 & 5 & 7 & 7 & 7 \\
\end{bmatrix}
\]

with respect to the standard bases. Find a basis for $\text{Ker}(T)$ and $\text{Im}(T)$.

Row-reducing gives:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 & 2 \\
\end{bmatrix}
\]

then swap these

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(pivots in column 1 & 2).

The homog. linear system thus has general solution

\[
\begin{bmatrix}
-x_5 \\
x_4 \\
-x_4 + x_5 \\
x_5 \\
\end{bmatrix}
\]

$(x_2, x_4, x_5$ free $)$

\[
= x_2 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

& the kernel has basis

\[
\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.
\]

The image has a basis given by the column w/ pivots in the reduced form, i.e. \[
\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.
\]
3. Let

\[ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]

The set \( \beta = \{ B_1, B_2, B_3, B_4 \} \) is a basis for \( M_{2\times 2}(\mathbb{R}) \). Let \( T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}) \) be given by \( T(A) = A^t - 2A \), where \( A^t \) denotes the transpose of \( A \). Take my word for it, that \( T \) is a linear transformation.

Find the matrix \( [T]_\beta \).

Note that for any matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), the coordinates in basis \( \beta \) are \( \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)_\beta = \begin{bmatrix} \frac{a}{c} \\ \frac{b}{d} \end{bmatrix} \) since \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

The columns of \( [T]_\beta \) are, respectively,

\[ [T(B_1)]_\beta = \left[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right] - 2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ [T(B_2)]_\beta = \left[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] - 2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \]

\[ [T(B_3)]_\beta = \left[ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right] - 2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \]

\[ [T(B_4)]_\beta = \left[ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right] - 2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \]

So putting them together,

\[ [T]_\beta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \]
4. Let $S : U \to V$ and $T : V \to W$ be linear transformations. Define the *composition* transformation $TS : U \to W$ by the equation $TS(\vec{u}) = T(S(\vec{u}))$ for all $\vec{u} \in U$. Prove that $TS$ is a linear transformation.

$$
\forall \vec{u}, \vec{v} \in U, \ c \in \mathbb{R},
TS(\vec{u} + c\vec{v}) = T(S(\vec{u} + c\vec{v}))
= T(S(\vec{u}) + cS(\vec{v})) \quad \text{(since $S$ is linear)}
= T(S(\vec{u})) + cT(S(\vec{v})) \quad \text{(since $T$ is linear)}
= TS(\vec{u}) + cTS(\vec{v}).
$$

so $TS$ is also a linear transformation.
5. Let $T : V \rightarrow W$ be an injective linear transformation. Prove that if \( \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq V \) is linearly independent, then \( \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\} \) is linearly independent.

Suppose that $c_1, \ldots, c_k \in \mathbb{R}$ satisfy

$$
\sum_{i=0}^{k} c_i T(\mathbf{v}_i) = \mathbf{0}.
$$

By linearity of $T$, this implies that

$$
T\left( \sum_{i=1}^{k} c_i \mathbf{v}_i \right) = \mathbf{0}.
$$

Since $T$ is injective, this implies that

$$
\sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{0},
$$

and finally since \( \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \) is a linearly independent set, this implies that

$$
c_1 = c_2 = \cdots = c_k = 0,
$$

which shows that \( \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\} \) is linearly independent.