Amherst College
Department of Mathematics

Math 271 | Midterm 2 Practice Exam 2 | Spring 2022

Name: Solutions

This is a modified version of a practice exam from Fall 2018.

Read This First!

- Please read each question carefully. Show **ALL** work clearly in the space provided.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question.
- Please cross out or fully erase any work that you do not want graded.
- The point value of each question is indicated after its statement.
- No books or other references are permitted.
- Calculators are not allowed and you must show all your work.

Grading - For Administrative Use Only

Question:	1	2	3	4	5	Total
Points:	15	15	15	15	15	75
Score:						

1. True or False: (No justification necessary.)

(a) Every injective linear transformation $T : V \to W$ is also surjective. **T** **F** [3]

(eg. $\mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (x,y,0)$)

(b) If V is a finite dimensional vector space, then there is a linear transformation $T : V \to V$ such that $[T]_\alpha^\alpha = [T]_\beta^\beta$ for all α and β bases of V. Comment (2022): this problem concerns a topic we haven't discussed much this semester, so I'd be unlikely to ask this question, but it may still be useful to try to figure it out. **T** **F** [3]

(eg. the identity transformation, or the zero transformation)

(c) If $T : P_5(\mathbb{R}) \to \mathbb{R}^5$ is linear, then T is not surjective. **T** **F** [3]

($\dim P_5(\mathbb{R}) > \dim \mathbb{R}^5$; surjective maps exist)

(d) If V is a finite-dimensional vector space and W is a subspace of V, then $\dim(V) \leq \dim(W)$. **T** **F** [3]

(instead, $\dim W \leq \dim V$)

2. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation with matrix representation [15]

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3_{-3} & 3_{-3} & 3_{-3} & 3_{-3} & 3_{-3} \\ 5_{-5} & 5_{-5} & 7_{-5} & 7_{-5} & 7_{-5} \end{bmatrix} \begin{matrix} \\ -3R1 \\ -5R1 \end{matrix}$$

with respect to the standard bases. Find a basis for Ker(T) and Im(T).

Row-reducing gives:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \end{pmatrix} \begin{matrix} -\tfrac{1}{2}R3 \\ \\ *1/2 \end{matrix} \Big\} \text{ then swap these}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(pivots in columns 1\&3)}$$

The homog. linear system thus has general solution

$$\begin{pmatrix} -x_2 \\ x_2 \\ -x_4-x_5 \\ x_4 \\ x_5 \end{pmatrix} \quad (x_2, x_4, x_5 \text{ free})$$

$$= x_2 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

& the kernel has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

The image has a basis given by the column w/ pivots in the reduced form, i.e. $\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \right\}$.

3. Let [15]

$$B_1 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \quad B_2 = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \quad B_3 = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \quad B_4 = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$

The set $\beta = \{B_1, B_2, B_3, B_4\}$ is a basis for $M_{2\times 2}(\mathbb{R})$. Let $T : M_{2\times 2}(\mathbb{R}) \to \mathrm{M}_{2\times 2}(\mathbb{R})$ be given by $T(A) = A^t - 2A$, where A^t denotes the transpose of A. Take my word for it, that T is a linear transformation.

Find the matrix $[T]_\beta^\beta$.

Note that for any matrix $\begin{pmatrix}a & b\\c & d\end{pmatrix}$, the coordinates in basis β are $\left[\begin{pmatrix}a & b\\c & d\end{pmatrix}\right]_\beta = \begin{pmatrix}a\\b\\c\\d\end{pmatrix}$ since $\begin{pmatrix}a & b\\c & d\end{pmatrix} = a\cdot\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$.

The columns of $[T]_\beta^\beta$ are, respectively,

$$[T(B_1)]_\beta = \left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}\right]_\beta = \begin{pmatrix}-1\\0\\0\\0\end{pmatrix}$$

$$[T(B_2)]_\beta = \left[\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} - \begin{pmatrix}0 & 2\\0 & 0\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\-2\\1\\0\end{pmatrix}$$

$$[T(B_3)]_\beta = \left[\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\2 & 0\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\1\\-2\\0\end{pmatrix}$$

$$[T(B_4)]_\beta = \left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 2\end{pmatrix}\right]_\beta = \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}$$

So putting these together,

$$[T]_\beta^\beta = \begin{pmatrix}-1 & 0 & 0 & 0\\0 & -2 & 1 & 0\\0 & 1 & -2 & 0\\0 & 0 & 0 & -1\end{pmatrix}.$$

4. Let $S : U \to V$ and $T : V \to W$ be linear transformations. Define the *composition* transformation $TS : U \to W$ by the equation $TS(\vec{u}) = T(S(\vec{u}))$ for all $\vec{u} \in U$. Prove that TS is a linear transformation. [15]

$\forall \vec{u}, \vec{v} \in U, \quad c \in \mathbb{R},$

$$\begin{aligned} TS(\vec{u} + c\vec{v}) &= T\left(S(\vec{u} + c\vec{v})\right) \\ &= T\left(S(\vec{u}) + c\,S(\vec{v})\right) \quad \text{(since } S \text{ is linear)} \\ &= T(S(\vec{u})) + cT(S(\vec{v})) \quad \text{(since } T \text{ is linear)} \\ &= TS(\vec{u}) + c\,TS(\vec{v}), \end{aligned}$$

so TS is also a linear transformation

5. Let $T : V \to W$ be an injective linear transformation. Prove that if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq V$ is linearly independent, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is linearly independent. [15]

Suppose that $c_1, \cdots, c_k \in \mathbb{R}$ satisfy

$$\sum_{i=0}^{k} c_i T(\vec{v}_i) = \vec{0}.$$

By linearity of T, this implies that

$$T\left(\sum_{i=1}^{k} c_i \vec{v}_i\right) = \vec{0}.$$

Since T is injective, this implies that

$$\sum_{i=1}^{k} c_i \vec{v}_i = \vec{0},$$

and finally since $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is a linearly independent set, this implies that

$$c_1 = c_2 = \cdots = c_k = 0,$$

which shows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_k)\}$ is linearly independent.