Math 271, Spring 2021

Extra Practice Problems for Exam 2

The problems and solutions below are gratefully borrowed, with minor modifications, from practice problems written by Rob Benedetto and Sema Gunturkun.

The following TRUE/FALSE questions provide you very good practice on understanding of the concepts overall.

1. Determine if each of following is **True** or **False**. If it is true, then give a short proof to justify your answer. If it is false, then either explain why clearly or give a precise counter example.

True / False A linear transformation $T : V \to W$ has a matrix $[T]_\alpha^\beta \in M_{4\times 5}(\mathbb{R})$ then $\dim V = 4$ and $\dim W = 5$.

True / False Let V, W be finite dimensional vector spaces such that $\dim V = 4$ and $\dim W = 3$. There is an injective (i.e 1-1) linear transformation $T : V \to W$ such that T is not the zero map.

True / False Let U, V be finite dimensional vector spaces such that $\dim V = 3$ and $\dim W = 4$. There is no surjective (i.e. onto) linear transformation $T : V \to W$ such that T is not the zero map.

True / False Let $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\alpha' = \{\vec{v}_3, \vec{v}_2, \vec{v}_1\}$ be (ordered) bases for V. (Notice they are the same sets but vectors ordered differently.) Then the matrix $[\mathrm{I}_V]_\alpha^{\alpha'}$ of the identity map $\mathrm{I}_V : V \to V$ is the identity matrix $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

True / False Let A be a 7×5 matrix then the largest possible rank of A is 7.

2. Let $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ be a map defined by $T(f) = (f(-2), f'(3))$.

(a) Prove that T is a linear transformation.

(b) Let $\beta = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$, and let $\gamma = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$. Compute the matrix $[T]_\beta^\gamma$.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $T(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}) = \begin{bmatrix} b \\ 2a + d \\ 3b \end{bmatrix}$.

(a) Find a basis for Ker(T) *“ the Kernel of T.*

(b) Find a basis for Im(T) *“ the Image of T.*

(c) What is the nullity of T?

(d) What is the rank of T?

4. The following maps are both linear. For each, decide whether or not it is an isomorphism. If you see a fast method, feel free to use it, but don’t forget to explain your reasoning.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^4$ by $T(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} 5a - b \\ 6b \\ 2a - 7b \\ 3a \end{bmatrix}$.

(b) $U : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $U(f) = f(x) - xf'(x) + 2f(3)$.

7. Let $\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ (i.e. the standard basis) and $\gamma = \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$.

(a) Find the vector $\vec{x} \in \mathbb{R}^2$ whose coordinate vector with respect to γ is $[\vec{x}]_\gamma = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(b) Find the following coordinate vectors with respect to the indicated basis.

(i) Find $[\vec{v}]_\alpha$ where $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

(ii) Find $[\vec{v}]_\gamma$ where $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

(iii) Find $[\vec{u}]_\gamma$ where $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$.

(i) Compute the matrix $[T]_\alpha^\alpha$.
(ii) Compute the matrix $[T]_\gamma^\alpha$.

8. Let V, W be vector spaces, let $T : V \to W$ be a linear map, let $\beta = \{\vec{v}_1, \dots, \vec{v}_n\} \subseteq V$ be a basis for V. Define γ to be $\gamma = \{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$, and suppose that $\text{Span}(\gamma) = W$. Prove that T is onto.

9. Let $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 3 & 3 & 3 & 6 \end{bmatrix}$. Let $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $T_A(\vec{x}) = A\vec{x}$.

(a) Find bases for the kernel $\text{Ker}(T_A)$ (a.k.a.$\text{Ker}(A)$) and image $\text{Im}(T_A)$ (a.k.a. $\text{Im}(A)$).

(b) What are the rank and nullity of A?

10. Let $A \in M_{3\times3}(\mathbb{R})$ be a 3×3 matrix such that the equation $A\vec{x} = \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}$ has exactly one solution.

Prove that for any $\vec{b} \in \mathbb{R}^3$, the system $A\vec{x} = \vec{b}$ is consistent and has exactly one solution.

11. Decide whether each of the following statements is True or False. A always denotes an $m \times n$ matrix, $\vec{b}$ a vector in $\mathbb{R}^m$ or $\mathbb{R}^n$, and $\vec{x}$ a (variable) vector in $\mathbb{R}^n$. (*Hint: For the below statements related to system of equations, you may think about the problems in terms of the linear transformation given as the multiplication by* A)

True / False	For any $\vec{b} \in \text{Im}(A)$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution.

True / False — For any $\vec{b} \in \text{Ker}(A)$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution.

True / False — If $\text{Im}(A) = \{\vec{0}\}$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution.

True / False — If $\text{Ker}(A) = \{\vec{0}\}$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution.

True / False — If $\text{rank}(A) = m$, then for ANY $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution.

True / False — If $\text{rank}(A) = n$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution.

12. Let $\alpha = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. be a basis for $\mathbb{R}^3$.

(a) Let $\vec{v}$ be the vector with α-coordinates $[\vec{v}]_\alpha = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Find the standard coordinates for $\vec{v}$ (i.e. the coordinate vector of $\vec{v}$ w.r.t. the standard basis.)

(b) Let $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Compute $[\vec{w}]_\alpha$.

13. Is it possible for a linear map $T : V \to W$ such that $\dim V = 3$ $\dim W = 5$ and $\text{rank}(T) = 4$? If so, write down an example of such a linear map and demonstrate that it has rank 4. If not, explain why such a linear map cannot exist.

14. Recall that $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a (standard) basis for $M_{2\times 2}(\mathbb{R})$, where

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear map defined by $T(A) = CA$, for any 2×2 matrix A.

(a) Find the matrix representing T with respect to the basis β. (That is, compute $[T]_\beta^\beta$.)

(b) Find a basis for $\text{Ker}(T)$.

(c) Find a basis for $\text{Im}(T)$.

15. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} \quad \text{and} \quad T(\begin{bmatrix} 2 \\ 2 \end{bmatrix}) = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

Compute $T(\begin{bmatrix} 6 \\ -1 \end{bmatrix})$.