Solutions to Extra Practice Problems for Exam 2

The problems and solutions below are gratefully borrowed, with minor modifications, from practice problems written by Rob Benedetto and Sema Gunturkun.

The following TRUE/FALSE questions provide you very good practice on understanding of the concepts overall.

1. Determine if each of following is True or False. If it is true, then give a short proof to justify your answer. If it is false, then either explain why clearly or give a precise counter example.

True / False A linear transformation $T : V \rightarrow W$ has a matrix $[T]_{\alpha}^{\beta} \in M_{4 \times 5}(\mathbb{R})$ then $\dim V = 4$ and $\dim W = 5$.

FALSE. If the matrix of $T$ is in $M_{4 \times 5}(\mathbb{R})$, it means dimension of target, i.e. $\dim W$ must be 4 and the domain $V$ must have dimension 5.

True / False Let $V, W$ be finite dimensional vector spaces such that $\dim V = 4$ and $\dim W = 3$. There is an injective (i.e 1-1) linear transformation $T : V \rightarrow W$ such that $T$ is not the zero map.

FALSE. There cannot be an injective because by Rank-Nullity Theorem;

$$\text{nullity}(T) + \text{rank}(T) = \dim V.$$ 

Since $\text{Im}(T) \subseteq W$, we get $\text{rank}(T) \leq 3$. Then $\text{nullity}(T) \geq 1$. Thus $\text{Ker}(T) \neq \{\vec{0}\}$.

True / False Let $U, V$ be finite dimensional vector spaces such that $\dim V = 3$ and $\dim W = 4$. There is no surjective (i.e. onto) linear transformation $T : V \rightarrow W$ such that $T$ is not the zero map.

TRUE. By Rank-Nullity Theorem; $\text{nullity}(T) + \text{rank}(T) = \dim V$. So $\text{rank}(T) = \dim \text{Im}(T) \leq 3 < \dim W = 4$. Thus, $\text{Im}(T)$ is always a proper subspace of $W$, that is such linear map $T$ is never surjective.

True / False Let $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\alpha' = \{\vec{v}_3, \vec{v}_2, \vec{v}_1\}$ be (ordered) bases for $V$. (Notice they are the same sets but vectors ordered differently.) Then the matrix $[I_{V}]_{\alpha}^{\alpha'}$ of the identity map $I_{V} : V \rightarrow V$ is the identity matrix $I_{3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

FALSE. $[I_{V}]_{\alpha}^{\alpha'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
2. Let $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^2$ be a map defined by $T(f) = (f(-2), f'(3))$.
   (a) Prove that $T$ is a linear transformation.
   (b) Let $\beta = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$, and let $\gamma = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$. Compute the matrix $[T]_{\beta}^{\gamma}$.

   **Answer.**
   (a) For all $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$ in $P_2(\mathbb{R})$:
   
   $T(p + q) = ((p + q)(-2), (p + q)'(3)) = (p(-2) + q(-2), p'(3) + q'(3)) = (p(-2), p'(3)) +
   (q(-2), q'(3)) = T(p) + T(q)$ (the second equation is using polynomial addition and derivative additive rule.)

   For all $p(x) \in P_2(\mathbb{R})$, and for all $c \in \mathbb{R}$,
   $T(cp) = ((cp)(-2), (cp)'(3)) = (cp(-2), cp'(3)) = c(p(-2), p'(3)) = cT(p)$ (similarly, the second equation is using scalar multiple of a polynomial and derivative rule for scalar multiple.)

   (b) We compute $T(1) = [1] = \vec{e}_1$, $T(x) = [-1] = -2\vec{e}_1 + \vec{e}_2$, and $T(x^2) = [2] = 4\vec{e}_1 + 6\vec{e}_2$.

   So the matrix is $[T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 6 \end{bmatrix}$.

3. Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a linear transformation defined by $T(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}) = \begin{bmatrix} b \\ 2a + d \\ 3b \end{bmatrix}$.

   (a) Find a basis for $\text{Ker}(T)$ “the Kernel of $T$.”
   (b) Find a basis for $\text{Im}(T)$ “the Image of $T$.”
   (c) What is the nullity of $T$?
   (d) What is the rank of $T$?

   (a) (Answer using definition of the Kernel of $T$) Setting $T(\begin{bmatrix} b \\ c \\ d \end{bmatrix}) = \vec{0}$ gives $b = 2a + d = 3b = 0$, or equivalently, $b = 0$ and $d = -2a$, and $c$ can be anything. So

   $\text{Ker}(T) = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ -2a \end{bmatrix} \mid a \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid a, c \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. 

   FALSE. By Rank-Nullity Theorem; nullity($A$) + rank($A$) = \#of columns of $A$ (as a linear transformation we can interpret this as $T : \mathbb{R}^5 \rightarrow \mathbb{R}^7$ given by $T(\vec{x}) = A\vec{x}$, then we have nullity($T$) + rank($T$) = dim($\mathbb{R}^5$)). Thus, rank($A$) could be at most 5 when nullity($A$) = 0.
Let \( \beta = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \} \). Then \( \beta \) is linearly independent, since it has two elements, and they are not scalar multiple of each other. Thus, \( \beta \) is a basis for \( \ker(T) \).

[Alternately (Using the matrix of \( T \)): \( T \) is multiplication by the matrix \( A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \end{bmatrix} \).

Doing row reduction to solve \( A \vec{x} = \vec{0} \) gives \( \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Then the solution set of \( A \vec{x} = \vec{0} \) has a basis \( \beta' = \{ \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \} \), which is also a basis for \( \ker(T) \subseteq \mathbb{R}^4 \). Notice that \( \text{Span}(\beta) = \text{Span}(\beta') \).]

(b) (Answer using definition of the Image of \( T \))

\[
\text{Im}(T) = \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^4 \} \\
= \{ \begin{bmatrix} 2a + d \\ 3b \end{bmatrix} : a, b, d \in \mathbb{R} \} \\
= \{ a \begin{bmatrix} 0 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} : a, b, d \in \mathbb{R} \} = \text{Span} \left( \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)
\]

Since \( \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), it is redundant and we can remove it from the spanning set. Hence, the set \( \gamma = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \} \) is linearly independent, since it has two elements, and they are not scalar multiple of each other. Thus, \( \gamma \) is a basis for \( \text{Im}(T) \).

[Alternately (Using the matrix of \( T \)): Again using the matrix \( A \) of linear transformation \( T \), the row-reduced form \( \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) tells us that first and second columns are pivot (therefore, linearly independent). So we take the first and second columns of \( A \); and we get \( \gamma' = \{ \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \} \), which is also a basis for \( \text{Im}(T) \subseteq \mathbb{R}^3 \). Notice that \( \text{Span}(\gamma) = \text{Span}(\gamma') \).]

(c) \( \text{rank}(T) = |\gamma| = 2 = \# \) linearly indep. columns of \( A \), that is, \( \# \) pivot columns of \( \text{rref}(A) \).

(d) \( \text{nullity}(T) = |\beta| = 2 = \# \) non-pivot columns of \( \text{rref}(A) \), that is, \( \# \) of free var. of \( A \vec{x} = \vec{0} \).

4. The following maps are both linear. For each, decide whether or not it is an isomorphism. If you see a fast method, feel free to use it, but don’t forget to explain your reasoning.

(a) \( T : \mathbb{R}^2 \to \mathbb{R}^4 \) by \( T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} 5a - b \\ 6b \\ 2a - 7b \\ 3a \end{bmatrix} \).

(b) \( U : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \) by \( U(f) = f(x) - xf'(x) + 2f(3) \).
Answer.
(a) Since \( \dim(\mathbb{R}^2) = 2 \neq 4 = \dim(\mathbb{R}^4) \), there cannot be an isomorphism between the two spaces, so \( T \) is not invertible.

(b) To find \( \ker(U) \) we solve \( U(a + bx + cx^2) = 0 \) then \( a + bx + cx^2 - (bx + 2cx^2) + 2(a + 3b + 9c) = 0 \), i.e., \( 3a + 6b + 18c - cx^2 = 0 \).
Thus, \( \ker(U) = \{ a + bx + cx^2 : a + 2b + 6c = -c = 0 \} \). Solving these equations gives \( c = 0 \) and \( a = -2b \). In particular, \( x - 2 \in \ker(U) \). [And a simple check shows that indeed, \( U(x - 2) = 0 \).]
So since \( \ker(U) \neq \{ \vec{0} \} \), \( U \) is not one-to-one, and hence \( U \) is not invertible.

[Alternately: One can construct the \( 3 \times 3 \) matrix of \( U \) and verify it is not invertible.]

7. Let \( \alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \) (i.e. the standard basis) and \( \gamma = \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\} \) be bases for \( \mathbb{R}^2 \).

(a) Find the vector \( \vec{x} \in \mathbb{R}^2 \) whose coordinate vector with respect to \( \gamma \) is \( [\vec{x}]_\gamma = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \).

(b) Find the following coordinate vectors with respect to the indicated basis.

(i) Find \( [\vec{v}]_\alpha \) where \( \vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \).

(ii) Find \( [\vec{v}]_\gamma \) where \( \vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \).

(iii) Find \( [\vec{u}]_\gamma \) where \( \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

(c) Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear map given by \( T(\vec{x}) = A\vec{x} \) where \( A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \).

(i) Compute the matrix \( [T]_\alpha^\alpha \).

(ii) Compute the matrix \( [T]_\gamma^\alpha \).

Answer.

(a) If the coordinate vector \( [\vec{x}]_\gamma = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) then it means we write \( \vec{x} = (-1) \begin{bmatrix} 5 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} \) as a linear combination of the vectors of basis \( \gamma \) (with respect to the given order). Thus, \( \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \).

(b) (i) \( \alpha = \{ \vec{e}_1, \vec{e}_2 \} \) is the standard basis for \( \mathbb{R}^2 \), so \( \vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5\vec{e}_1 + 3\vec{e}_2 \). Therefore, the coordinate vector \( [\vec{v}]_\alpha \) is \( \begin{bmatrix} 5 \\ 3 \end{bmatrix} \) itself.

(ii) Notice that the vector \( \vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \) is the first vector of \( \gamma \), so \( \vec{v} = 1 \begin{bmatrix} 5 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 1 \end{bmatrix} \). Thus, \( [\vec{v}]_\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).
(iii) Find \( \vec{u}_\gamma \) where \( \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

Write \( \vec{v} = c_1 \begin{bmatrix} 5 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} \) for (uniquely determined) \( c_1, c_2 \in \mathbb{R} \).

(We need to find \( c_1, c_2 \) b/c \( \vec{u}_\gamma = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \).)

Then
\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix}
\]
gives
\[
\begin{aligned}
5c_1 + 4c_2 &= 1 \\
3c_1 + 2c_2 &= 1
\end{aligned}
\]

(I skipped the details). Then we get \( c_1 = 1 \) and \( c_2 = -1 \). Hence \( \vec{u}_\gamma = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

(c) Recall that \( T(\vec{x}) = A\vec{x} \) where

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}
\]

(i) Since \( \alpha \) is the standard basis, we actually have \( A = [T]^\alpha_\alpha \) “the standard matrix of \( T \”).

(Convince yourselves by finding \( T(\vec{e}_1) \) and \( T(\vec{e}_2) \). I skip the details.)

(ii) To find \([T]^\alpha_\gamma\),

\[
T \begin{bmatrix} 5 \\ 3 \end{bmatrix} = A \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } T \begin{bmatrix} 4 \\ 2 \end{bmatrix} = A \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} .
\]

Hence, we get
\[
[T]^\alpha_\gamma = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}.
\]

8. Let \( V, W \) be vector spaces, let \( T : V \rightarrow W \) be a linear map, let \( \beta = \{ \vec{v}_1, \ldots, \vec{v}_n \} \subseteq V \) be a basis for \( V \). Define \( \gamma \) to be \( \gamma = \{ T(\vec{v}_1), \ldots, T(\vec{v}_n) \} \), and suppose that \( \text{Span}(\gamma) = W \). Prove that \( T \) is onto.

**Proof.** Let \( \vec{w} \in W \). Since \( \gamma = T(\beta) \) and \( \text{Span}(T(\beta)) = W \) is given, there exist scalars \( a_1, \ldots, a_n \in \mathbb{R} \) such that
\[
\vec{w} = a_1 T(\vec{v}_1) + \cdots + a_n T(\vec{v}_n) .
\]

Thus, \( \vec{w} = a_1 T(\vec{v}_1) + \cdots + a_n T(\vec{v}_n) = T(a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n) \). Hence, \( \vec{w} \) is in the image of \( T \), as desired, so \( W \subseteq \text{Im}(T) \). Thus, \( \text{Im}(T) = W \) (as we already have other side of the inclusion \( \supseteq \)).

QED

9. Let \( A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 3 & 3 & 3 & 6 \end{bmatrix} \). Let \( T_A : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) be given by \( T_A(\vec{x}) = A\vec{x} \).

(a) Find bases for the kernel \( \text{Ker}(T_A) \) (a.k.a.\( \ker(A) \)) and image \( \text{Im}(T_A) \) (a.k.a. \( \text{Im}(A) \)).

(b) What are the rank and nullity of \( A \)?

**Answer.**

(a) By Gauss-Jordan elimination, we get
\[
A \rightarrow R_2 \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 3 & 3 & 3 & 6 \end{bmatrix} \rightarrow R_1 \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 3 & 3 & 3 & 6 \end{bmatrix} \rightarrow R_3 - 3R_1 \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow R_2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)
\]
For Ker(A): The first row of the echelon form says $x_1 = -x_2$, and the second says $x_3 = -2x_4$, and $x_2 = t, x_4 = s$ are free. So $\text{Ker}(A) = \left\{ \begin{bmatrix} -t \\ t \\ -2s \\ s \end{bmatrix} : t, s \in \mathbb{R} \right\}$. Thus, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for Ker(A).

For Im(A), the first and third columns of the row-reduced echelon form are the pivot columns, so choosing the corresponding columns of $A$, we have $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ is a basis for Im(A).

(b) Counting elements of bases from part (a), the rank is 2, and the nullity is 2.

10. Let $A \in M_{3 \times 3}(\mathbb{R})$ be a $3 \times 3$ matrix such that the equation $A\vec{x} = \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}$ has exactly one solution. Prove that for any $\vec{b} \in \mathbb{R}^3$, the system $A\vec{x} = \vec{b}$ is consistent and has exactly one solution.

**Proof.** Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\vec{x}) = A\vec{x}$. If $A\vec{x} = \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}$ has exactly one solution $\vec{w}$, then it means that the inverse image $T^{-1}(\{ \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix} \}) = \{ \vec{w} \}$. On the other hand, we know that

$T^{-1}(\{ \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix} \}) = \{ \vec{w} + \vec{v}_h \mid \vec{v}_h \in \text{Ker}(T) \} = \{ \vec{w} + \vec{v}_h \mid \vec{v}_h \text{ is a solution for homogeneous } A\vec{x} = \vec{0} \}.$

Thus, $\text{Ker}(T) = \{ \vec{0} \}$, that is, $\text{nullity}(A) = 0$ (i.e. $T$ is injective) and then $\text{rank}(A) = 3 = \text{dim} \mathbb{R}^3$ (i.e. $T$ is surjective). Therefore, $T$ is an isomorphism. Then for any $\vec{b} \in \mathbb{R}^3$, there exist unique $\vec{y} \in \mathbb{R}^3$ such that $T(\vec{y}) = A\vec{y} = \vec{b}$. QED

11. Decide whether each of the following statements is True or False. $A$ always denotes an $m \times n$ matrix, $\vec{b}$ a vector in $\mathbb{R}^m$ or $\mathbb{R}^n$, and $\vec{x}$ a (variable) vector in $\mathbb{R}^n$. (Hint: For the below statements related to system of equations, you may think about the problems in terms of the linear transformation given as the multiplication by $A$)

| True / False | For any $\vec{b} \in \text{Im}(A)$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution. | TRUE |
| True / False | For any $\vec{b} \in \text{Ker}(A)$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution. | FALSE |
| True / False | If $\text{Im}(A) = \{ \vec{0} \}$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution. | FALSE |
| True / False | If $\text{Ker}(A) = \{ \vec{0} \}$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution. | TRUE |
| True / False | If $\text{rank}(A) = m$, then for ANY $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution. | TRUE |
If rank($A$) = $n$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution.

TRUE

12. Let $\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^3$.

(a) Let $\vec{v}$ be the vector with $\alpha$-coordinates $[\vec{v}]_\alpha = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Find the standard coordinates for $\vec{v}$ (i.e. the coordinate vector of $\vec{v}$ w.r.t. the standard basis.)

(b) Let $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Compute $[\vec{w}]_\alpha$.

Answer.

(a) Write $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Then $\vec{v} = -\vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 = \begin{bmatrix} -1 + 1 + 2 \\ 1 + 1 - 2 \\ 0 + 0 + 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

(b) We are solving $\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \end{bmatrix}$ to find the coefficients $x_1, x_2, x_3$ in $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ where $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ will be the $\alpha$-coordinate vector of $\vec{w}$. A quick Gauss-Jordan elimination (I skip the details of row-reduction here) gives $[\vec{w}]_\alpha = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$.

13. Is it possible for a linear map $T : V \to W$ such that $\dim V = 3 \dim W = 5$ and rank($T$) = 4? If so, write down an example of such a linear map and demonstrate that it has rank 4. If not, explain why such a linear map cannot exist.

Answer. NO. By Rank-Nullity theorem, we know that rank $T$ + nullity $T = \dim V = 3$. So rank $T = 3$ - nullity $T$ so the rank can be at most 3 (maximum rank occur when the nullity nullity $T = 0$). Thus, the rank cannot be 4.

14. Recall that $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a (standard) basis for $M_2\times_2(\mathbb{R})$, where

$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Let $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Let $T : M_2\times_2(\mathbb{R}) \to M_2\times_2(\mathbb{R})$ be the linear map defined by $T(A) = CA$, for any $2 \times 2$ matrix $A$.

(a) Find the matrix representing $T$ with respect to the basis $\beta$. (That is, compute $[T]_\beta$.)
(b) Find a basis for Ker(T).

(c) Find a basis for Im(T).

Answer.

(a) Computation shows \( T(E_{11}) = CE_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = E_{11} + E_{21} \). So the first column of \([T]_\beta\) is the coordinate vector of \( T(E_{11}) \) with respect to \( \beta \); \( [T(E_{11})]_\beta = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \). Similarly computing the other columns \([T(E_{12})]_\beta, [T(E_{21})]_\beta, [T(E_{22})]_\beta\) (in the given order of \( \beta \)), we get

\[
[T]_\beta = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.
\]

(b) Row-reducing \( A = [T]_\beta \) we get \( \text{rref}(A) = \text{rref}([T]_\beta) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). A basis for the solution set of \( A\vec{x} = \vec{0} \) is

\[
\begin{cases}
-2 \\ 0 \\ 0 \\ 0
\end{cases},
\begin{cases}
0 \\ 1 \\ 0 \\ 0
\end{cases}.
\]

Thus, a basis elements of Ker(T) is \( \{B_1, B_2\} \) where

\[
B_1 = (-2)E_{11} + 0E_{12} + 1E_{21} + 0E_{22} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and}
\]

\[
B_2 = 0E_{11} + (-2)E_{12} + 0E_{21} + 1E_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
\]

(c) We determine that the first and the second columns are pivot columns of \( A \) (as those are the pivot columns of the \( \text{rref}(A) \)). Then a basis for \( \text{Im}(T) = \{D_1, D_2\} \) where the coordinate vectors are

\[
[D_1]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{which is the first column of } A, \quad \text{and } [D_2]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{which is the second column of } A.
\]

Therefore,

\[
D_1 = 1E_{11} + 0E_{12} + 1E_{21} + 0E_{22} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and}
\]

\[
D_2 = 0E_{11} + 1E_{12} + 0E_{21} + 1E_{22} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.
\]

[Alternate solution for parts (b) and (d) via definitions of kernel and image. One can describe the basis for Kernel and Image by finding a spanning set for each using the transformation

\[
T(\begin{bmatrix} a \\ c \end{bmatrix}) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} a + 2c \\ b + 2d \end{bmatrix}
\]

as in Problem # 2. ]
(d) (i) Let \( \gamma = \{ M_1, M_2, M_3, M_4 \} \) (It must have 4 elements in it as the space \( M_{2 \times 2} \) is 4 dimensional). Then we know that the \( j \)-th column of the change of basis matrix \([I]_\gamma^\beta = [M_j]_\beta\), that is, the coordinate vector of \( M_j \) with respect to the standard basis \( \beta \).

Therefore, we get
\[
\gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}
\]

(ii) \([I]_\beta^\gamma = ([I]_\gamma^\beta)^{-1}\). (I skip the details here). Then
\[
[I]_\beta^\gamma = \begin{bmatrix}
1 & -1 & -1 & 0 \\
-1/2 & 1 & 1 & 1/2 \\
1/2 & 0 & 0 & -1/2 \\
-1/2 & 0 & 1 & 1/2 \\
\end{bmatrix}
\]

(iii) \([T]_\gamma^\beta = [I]_\beta^\gamma [T]_\beta^\gamma [I]_\gamma^\beta = [T]_\beta^\gamma [I]_\gamma^\beta\).

Therefore,
\[
[T]_\gamma^\beta = \begin{bmatrix}
1 & 1 & 3 & 2 \\
2 & 3 & -2 & -1 \\
1 & 1 & 3 & 2 \\
2 & 3 & -2 & -1 \\
\end{bmatrix}
\]

For practice, verify that the columns of \([T]_\gamma^\beta\) are, indeed, the \( \beta \)-coordinate vectors of the images of \( M_i \in \gamma \), \([T(M_i)]_\beta\), for each \( i = 1, 2, 3, 4 \).

(iv) \([T]_\gamma^\gamma = [I]_\beta^\gamma [T]_\beta^\gamma [I]_\gamma^\beta\) (which can be also observed as \([T]_\gamma^\gamma = Q^{-1}[T]_\beta^\gamma Q\) where \( Q = [I]_\gamma^\beta \).

Then we get
\[
[T]_\gamma^\gamma = \begin{bmatrix}
-2 & -3 & 2 & 1 \\
7/2 & 5 & -3/2 & -1/2 \\
-1/2 & -1 & 5/2 & 3/2 \\
3/2 & 2 & 1/2 & 1/2 \\
\end{bmatrix}
\]

15. Let \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) be a linear transformation such that
\[
T\left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} \quad \text{and} \quad T\left( \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}
\]

Compute \( T\left( \begin{bmatrix} 6 \\ -1 \end{bmatrix} \right) \).

Answer. (Notice that \( T \) is given on a basis for \( \mathbb{R}^2 \).) We first need to write \( \begin{bmatrix} 6 \\ -1 \end{bmatrix} \) as a linear combination of \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 2 \end{bmatrix} \). That is, solve \( x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \), which gives the system \[
\begin{bmatrix}
2 & 2 & | & 6 \\
1 & 2 & | & -1 \\
\end{bmatrix}
\]

Row reduction (detailed omitted here) leads to \[
\begin{bmatrix}
1 & 0 & | & 7 \\
0 & 1 & | & -4 \\
\end{bmatrix}
\]

i.e., \( x = 7 \), \( y = -4 \).
A quick check shows $7 \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$. Thus, by linearity of $T$ we get

$T(\begin{bmatrix} 6 \\ -1 \end{bmatrix}) = T(7 \begin{bmatrix} 2 \\ 2 \end{bmatrix}) = 7T(2, 1) - 4T(2, 2) = 7 \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ -21 \end{bmatrix}.$