Math 271, Spring 2021

Solutions to Extra Practice Problems for Exam 2

The problems and solutions below are gratefully borrowed, with minor modifications, from practice problems written by Rob Benedetto and Sema Gunturkun.

The following TRUE/FALSE questions provide you very good practice on understanding of the concepts overall.

1. Determine if each of following is **True** or **False**. If it is true, then give a short proof to justify your answer. If it is false, then either explain why clearly or give a precise counter example.

True / False A linear transformation $T : V \to W$ has a matrix $[T]_\alpha^\beta \in M_{4\times 5}(\mathbb{R})$ then $\dim V = 4$ and $\dim W = 5$.

FALSE. if the matrix of T is in $M_{4\times 5}(\mathbb{R})$, it means dimension of target, i.e. $\dim W$ must be 4 and the domain V must have dimension 5.

True / False Let V, W be finite dimensional vector spaces such that $\dim V = 4$ and $\dim W = 3$. There is an injective (i.e 1-1) linear transformation $T : V \to W$ such that T is not the zero map.

FALSE. There cannot be an injective because by Rank-Nullity Theorem;

$$\text{nullity}(T) + \text{rank}(T) = \underbrace{\dim V}_{4}.$$

Since $\text{Im}(T) \subseteq W$, we get $\text{rank}(T) \leq 3$. Then $\text{nullity}(T) \geq 1$. Thus $\text{Ker}(T) \neq \{\vec{0}\}$.

True / False Let U, V be finite dimensional vector spaces such that $\dim V = 3$ and $\dim W = 4$. There is no surjective (i.e. onto) linear transformation $T : V \to W$ such that T is not the zero map.

TRUE. By Rank-Nullity Theorem; $\text{nullity}(T) + \text{rank}(T) = \underbrace{\dim V}_{3}$. So $\text{rank}(T) = \dim \text{Im}(T) \leq 3 < \dim W = 4$. Thus, $\text{Im}(T)$ is always a **proper** subspace of W, that is such linear map T is never surjective.

True / False Let $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\alpha' = \{\vec{v}_3, \vec{v}_2, \vec{v}_1\}$ be (ordered) bases for V. (Notice they are the same sets but vectors ordered differently.) Then the matrix $[\text{I}_V]_\alpha^{\alpha'}$ of the identity map $\text{I}_V : V \to V$ is the identity matrix $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

FALSE. $[\text{I}_V]_\alpha^{\alpha'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

True / False Let A be a 7×5 matrix then the largest possible rank of A is 7.

FALSE. By Rank-Nullity Theorem; $\text{nullity}(A) + \text{rank}(A) = \underbrace{\#\text{of columns of } A}_{=5}$ (as a linear transformation we can interpret this as $T : \mathbb{R}^5 \to \mathbb{R}^7$ given by $T(\vec{x}) = A\vec{x}$, then we have $\text{nullity}(T) + \text{rank}(T) = \dim \mathbb{R}^5$). Thus, $\text{rank}(A)$ could be at most 5 when $\text{nullity}(A) = 0$.

2. Let $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ be a map defined by $T(f) = (f(-2), f'(3))$.

(a) Prove that T is a linear transformation.

(b) Let $\beta = \{1, x, x^2\}$ be the standard basis for $P_2(\mathbb{R})$, and let $\gamma = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$. Compute the matrix $[T]_\beta^\gamma$.

Answer.

(a) For all $p(x) = a_0 + a_1x + a_2x^2$ and $q(x) = b_0 + b_1x + b_2x^2$ in $P_2(\mathbb{R})$;
$T(p+q) = \big((p+q)(-2), (p+q)'(3)\big) = \big(p(-2) + q(-2), p'(3) + q'(3)\big) = \big(p(-2), p'(3)\big) + \big(q(-2), q'(3)\big) = T(p) + T(q)$ (the second equation is using polynomial addition and derivative additive rule.)

For all $p(x) \in P_2(\mathbb{R})$, and for all $c \in \mathbb{R}$,
$T(cp) = \big((cp)(-2), (cp)'(3)\big) = \big(cp(-2), cp'(3)\big) = c\big(p(-2), p'(3)\big) = cT(p)$ (similarly, the second equation is using scalar multiple of a polynomial and derivative rule for scalar multiple.)

(b) We compute $T(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{e}_1$, $T(x) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2\vec{e}_1 + \vec{e}_2$, and $T(x^2) = \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 4\vec{e}_1 + 6\vec{e}_2$.

So the matrix is $[T]_\beta^\gamma = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & 6 \end{bmatrix}$.

3. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $T(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}) = \begin{bmatrix} b \\ 2a + d \\ 3b \end{bmatrix}$.

(a) Find a basis for $\text{Ker}(T)$ " *the Kernel of T*.

(b) Find a basis for $\text{Im}(T)$ " *the Image of T*.

(c) What is the nullity of T?

(d) What is the rank of T?

(a) **(Answer using definition of the Kernel of T)** Setting $T(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}) = \vec{0}$ gives $b = 2a + d = 3b = 0$, or equivalently, $b = 0$ and $d = -2a$, and c can be anything. So

$$\text{Ker}(T) = \{\begin{bmatrix} a \\ 0 \\ c \\ -2a \end{bmatrix} \mid a, c \in \mathbb{R}\} \qquad = \{a\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mid a, c \in \mathbb{R}\} = \text{Span}\,(\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\}).$$

Let $\beta = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \}$. Then β is linearly independent, since it has two elements, and they are not scalar multiple of each other. Thus, β is a basis for $\text{Ker}(T)$.

[**Alternately (Using the matrix of T):** T is multiplication by the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \end{bmatrix}$. Doing row reduction to solve $A\vec{x} = \vec{0}$ gives $rref(A) = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then the solution set of $A\vec{x} = \vec{0}$ has a basis $\beta' = \{ \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \}$, which is also a basis for $\text{Ker}(T) \subseteq \mathbb{R}^4$. Notice that $\text{Span}(\beta) = \text{Span}(\beta')$.]

(b) **(Answer using definition of the Image of T)**

$$\begin{aligned} \text{Im}(T) &= \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^4\} \\ &= \{ \begin{bmatrix} b \\ 2a+d \\ 3b \end{bmatrix} : a, b, d \in \mathbb{R} \} \\ &= \{ a \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : a, b, d \in \mathbb{R} \} = \text{Span}\left(\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} \right) \end{aligned}$$

Since $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, it is redundant and we can remove it from the spanning set. Hence, the set $\gamma = \{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$ is linearly independent, since it has two elements, and they are not scalar multiple of each other. Thus, γ is a basis for $\text{Im}(T)$.

[**Alternately(Using the matrix of T):** Again using the matrix A of linear transformation T, the row-reduced form $rref(A) = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ tells us that first and second columns are pivot (therefore, linearly independent). So we take the first and second columns of A; and we get $\gamma' = \{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \}$, which is also a basis for $\text{Im}(T) \subseteq \mathbb{R}^3$. Notice that $\text{Span}(\gamma) = \text{Span}(\gamma')$.]

(c) $\text{rank}(T) = |\gamma| = 2 = \#$ linearly indep. columns of A, that is, # pivot columns of $\text{rref}(A)$.

(d) $\text{nullity}(T) = |\beta| = 2 = \#$ non-pivot columns of $\text{rref}(A)$, that is, # of free var. of $A\vec{x} = \vec{0}$.

4. The following maps are both linear. For each, decide whether or not it is an isomorphism. If you see a fast method, feel free to use it, but don't forget to explain your reasoning.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^4$ by $T(\begin{bmatrix} a \\ b \end{bmatrix}) = \begin{bmatrix} 5a - b \\ 6b \\ 2a - 7b \\ 3a \end{bmatrix}$.

(b) $U : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ by $U(f) = f(x) - xf'(x) + 2f(3)$.

Answer.

(a) Since $\dim(\mathbb{R}^2) = 2 \neq 4 = \dim(\mathbb{R}^4)$, there cannot be an isomorphism between the two spaces, so T is not invertible.

(b) To find $\mathrm{Ker}(U)$ we solve $U(a+bx+cx^2) = 0$ then $a+bx+cx^2-(bx+2cx^2)+2(a+3b+9c) = 0$, i.e., $(3a+6b+18c)-cx^2 = 0$.

Thus, $\mathrm{Ker}(U) = \{a+bx+cx^2 : a+2b+6c = -c = 0\}$. Solving these equations gives $c = 0$ and $a = -2b$. In particular, $x-2 \in \mathrm{Ker}(U)$. [And a simple check shows that indeed, $U(x-2) = 0$.]

So since $\mathrm{Ker}(U) \neq \{\vec{0}\}$, U is not one-to-one, and hence U is not invertible.

[Alternately: One can construct the 3×3 matrix of U and verify it is not invertible.]

7. Let $\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ (i.e. the standard basis) and $\gamma = \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$.

(a) Find the vector $\vec{x} \in \mathbb{R}^2$ whose coordinate vector with respect to γ is $[\vec{x}]_\gamma = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(b) Find the following coordinate vectors with respect to the indicated basis.

(i) Find $[\vec{v}]_\alpha$ where $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

(ii) Find $[\vec{v}]_\gamma$ where $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

(iii) Find $[\vec{u}]_\gamma$ where $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$.

(i) Compute the matrix $[T]_\alpha^\alpha$.

(ii) Compute the matrix $[T]_\gamma^\alpha$.

Answer.

(a) If the coordinate vector $[\vec{x}]_\gamma = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, then it means we write $\vec{x} = (\mathbf{-1}).\begin{bmatrix} 5 \\ 3 \end{bmatrix} + \mathbf{2}.\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ as a linear combination of the vectors of basis γ (with respect to the given order). Thus, $\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) (i) $\alpha = \{\vec{e}_1, \vec{e}_2\}$ is the standard basis for $\mathbb{R}^2$, so $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = 5\vec{e}_1 + 3\vec{e}_2$. Therefore, the coordinate vector $[\vec{v}]_\alpha$ is $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ itself.

(ii) Notice that the vector $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is the first vector of γ, so $\vec{v} = 1.\begin{bmatrix} 5 \\ 3 \end{bmatrix} + 0.\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Thus, $[\vec{v}]_\gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(iii) Find $[\vec{u}]_\gamma$ where $\vec{u} = \begin{bmatrix}1\\1\end{bmatrix}$.

Write $\vec{v} = c_1\begin{bmatrix}5\\3\end{bmatrix} + c_2\begin{bmatrix}4\\2\end{bmatrix}$ for (uniquely determined) $c_1, c_2 \in \mathbb{R}$.

(We need to find c_1, c_2 b/c $[\vec{u}]_\gamma = \begin{bmatrix}c_1\\c_2\end{bmatrix}$.)

Then $\begin{bmatrix}1\\1\end{bmatrix} = c_1\begin{bmatrix}5\\3\end{bmatrix} + c_2\begin{bmatrix}4\\2\end{bmatrix}$ gives

$$\begin{aligned}5c_1 + 4c_2 &= 1\\ 3c_1 + 2c_2 &= 1\end{aligned}$$

(I skipped the details). Then we get $c_1 = 1$ and $c_2 = -1$. Hence $[\vec{u}]_\gamma = \begin{bmatrix}1\\-1\end{bmatrix}$.

(c) Recall that $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}$.

(i) Since α is the standard basis, we actually have $A = [T]_\alpha^\alpha$ "*the standard matrix of* T". (Convince yourselves by finding $T(\vec{e}_1)$ and $T(\vec{e}_2)$. I skip the details.)

(ii) To find$[T]_\gamma^\alpha$;

$T(\begin{bmatrix}5\\3\end{bmatrix}) = A.\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$ and $T(\begin{bmatrix}4\\2\end{bmatrix}) = A.\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$.

Hence, we get

$[T]_\gamma^\alpha = \begin{bmatrix}3 & 2\\1 & 0\end{bmatrix}$.

8. Let V, W be vector spaces, let $T : V \to W$ be a linear map, let $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ be a basis for V. Define γ to be $\gamma = \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$, and suppose that $\text{Span}(\gamma) = W$. Prove that T is onto.

Proof. Let $\vec{w} \in W$. Since $\gamma = T(\beta)$ and $\text{Span}(T(\beta)) = W$ is given, there exist scalars $a_1, \ldots, a_n \in \mathbb{R}$ such that
$\vec{w} = a_1T(\vec{v}_1) + \cdots + a_nT(\vec{v}_n)$. Thus, $\vec{w} = a_1T(\vec{v}_1) + \cdots + a_nT(\vec{v}_n) = T(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n)$. Hence, $\vec{w}$ is in the image of T, as desired, so $W \subseteq \text{Im}(T)$. Thus, $\text{Im}(T) = W$ (as we already have other side of the inclusion $\supseteq$.) QED

9. Let $A = \begin{bmatrix}0 & 0 & 1 & 2\\1 & 1 & 2 & 4\\3 & 3 & 3 & 6\end{bmatrix}$. Let $T_A : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $T_A(\vec{x}) = A\vec{x}$.

(a) Find bases for the kernel $\text{Ker}(T_A)$ (a.k.a.$\text{Ker}(A)$) and image $\text{Im}(T_A)$ (a.k.a. $\text{Im}(A)$).

(b) What are the rank and nullity of A?

Answer.

(a) By Gauss-Jordan elimination, we get

$$A \to \begin{matrix}R_2\\R_1\\ \\ \end{matrix}\begin{bmatrix}1 & 1 & 2 & 4\\0 & 0 & 1 & 2\\3 & 3 & 3 & 6\end{bmatrix} \to \begin{matrix}\\ \\ \frac{R_3 - 3R_1}{-3}\end{matrix}\begin{bmatrix}1 & 1 & 2 & 4\\0 & 0 & 1 & 2\\0 & 0 & 1 & 2\end{bmatrix} \to \begin{matrix}R_1 - 2R_2\\ \\ R_3 - R_2\end{matrix}\begin{bmatrix}1 & 1 & 0 & 0\\0 & 0 & 1 & 2\\0 & 0 & 0 & 0\end{bmatrix} = rref(A)$$

For Ker(A): The first row of the echelon form says $x_1 = -x_2$, and the second says $x_3 = -2x_4$, and $x_2 = t$, $x_4 = s$ are free. So $\text{Ker}(A) = \left\{ \begin{bmatrix} -t \\ t \\ -2s \\ s \end{bmatrix} : t, s \in \mathbb{R} \right\}$. Thus, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for Ker(A).

For Im(A), the first and third columns of the row-reduced echelon form are the pivot columns, so choosing the corresponding columns of A, we have $\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a basis for Im(A).

(b) Counting elements of bases from part (a), the rank is 2, and the nullity is 2.

10. Let $A \in M_{3\times 3}(\mathbb{R})$ be a 3×3 matrix such that the equation $A\vec{x} = \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}$ has exactly one solution. Prove that for any $\vec{b} \in \mathbb{R}^3$, the system $A\vec{x} = \vec{b}$ is consistent and has exactly one solution.

Proof. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\vec{x}) = A\vec{x}$. If $A\vec{x} = \begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}$ has exactly one solution $\vec{w}$, then it means that the inverse image $T^{-1}(\{\begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}\}) = \{\vec{w}\}$.
On the other hand, we know that

$$T^{-1}(\{\begin{bmatrix} 5 \\ -7 \\ 0 \end{bmatrix}\}) = \{\vec{w} + \vec{v}_h \mid \vec{v}_h \in \text{Ker}(T)\} = \{\vec{w} + \vec{v}_h \mid \vec{v}_h \text{ is a solution for homogeneous } A\vec{x} = \vec{0}\}.$$

Thus, $\text{Ker}(T) = \{\vec{0}\}$, that is, $\text{nullity}(A) = 0$ (i.e. T is injective) and then $\text{rank}(A) = 3 = \dim \mathbb{R}^3$ (i.e. T is surjective). Therefore, T is an isomorphism. Then for any $\vec{b} \in \mathbb{R}^3$, there exist unique $\vec{y} \in \mathbb{R}^3$ such that $T(\vec{y}) = A\vec{y} = \vec{b}$. QED

11. Decide whether each of the following statements is True or False. A always denotes an $m \times n$ matrix, $\vec{b}$ a vector in $\mathbb{R}^m$ or $\mathbb{R}^n$, and $\vec{x}$ a (variable) vector in $\mathbb{R}^n$. (*Hint: For the below statements related to system of equations, you may think about the problems in terms of the linear transformation given as the multiplication by* A)

True / False — For any $\vec{b} \in \text{Im}(A)$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution.
TRUE

True / False — For any $\vec{b} \in \text{Ker}(A)$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution.
FALSE

True / False — If $\text{Im}(A) = \{\vec{0}\}$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution.
FALSE

True / False — If $\text{Ker}(A) = \{\vec{0}\}$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution.
TRUE

True / False — If $\text{rank}(A) = m$, then for ANY $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has AT LEAST ONE solution.
TRUE

True / False If $\text{rank}(A) = n$, then the equation $A\vec{x} = \vec{0}$ has AT MOST ONE solution.

TRUE

12. Let $\alpha = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. be a basis for $\mathbb{R}^3$.

(a) Let $\vec{v}$ be the vector with α-coordinates $[\vec{v}]_\alpha = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Find the standard coordinates for $\vec{v}$ (i.e. the coordinate vector of $\vec{v}$ w.r.t. the standard basis.)

(b) Let $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Compute $[\vec{w}]_\alpha$.

Answer.

(a) Write $\alpha = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Then $\vec{v} = -\vec{v}_1 + \vec{v}_2 + 2\vec{v}_3 = \begin{bmatrix} -1+1+2 \\ 1+1-2 \\ 0+0+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

(b) We are solving $\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ to find the coefficients x_1, x_2, x_3 in $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$ where $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ will be the α-coordinate vector of $\vec{w}$. A quick Gauss-Jordan elimination (I skip the details of row-reduction here) gives $[\vec{w}]_\alpha = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$.

13. Is it possible for a linear map $T : V \to W$ such that $\dim V = 3$ $\dim W = 5$ and $\text{rank}(T) = 4$? If so, write down an example of such a linear map and demonstrate that it has rank 4. If not, explain why such a linear map cannot exist.

Answer. NO. By Rank-Nullity theorem, we know that $\text{rank}\, T + \text{nullity}\, T = \dim V = 3$. So $\text{rank}\, T = 3 - \text{nullity}\, T$ so the rank can be at most 3 (maximum rank occur when the nullity $\text{nullity}\, T = 0$). Thus,the rank cannot be 4.

14. Recall that $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a (standard) basis for $M_{2\times 2}(\mathbb{R})$, where

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $C = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear map defined by $T(A) = CA$, for any 2×2 matrix A.

(a) Find the matrix representing T with respect to the basis β. (That is, compute $[T]_\beta^\beta$.)

(b) Find a basis for Ker(T).

(c) Find a basis for Im(T).

Answer.

(a) Computation shows $T(E_{11}) = CE_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = E_{11} + E_{21}$. So the first column of $[T]_\beta^\beta$ is the coordinate vector of $T(E_{11})$ with respect to β; $[T(E_{11})]_\beta = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Similarly computing the other columns $[T(E_{12})]_\beta, [T(E_{21})]_\beta, [T(E_{22})]_\beta$ (in the given order of β), we get

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

(b) Row-reducing $A = [T]_\beta^\beta$ we get $rref(A) = rref([T]_\beta^\beta) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A basis for the solution set of $A\vec{x} = \vec{0}$ is $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$.Thus, a basis elements of Ker(T) is $\{B_1, B_2\}$ where

$$B_1 = (-2)E_{11} + 0E_{12} + 1E_{21} + 0E_{22} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and}$$

$$B_2 = 0E_{11} + (-2)E_{12} + 0E_{21} + 1E_{22} = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}.$$

(c) We determine that the first and the second columns are pivot columns of A (as those are the pivot columns of the $rref(A)$). Then a basis for Im(T) $= \{D_1, D_2\}$ where the coordinate vectors are $[D_1]_\beta = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, which is the first column of A, and $[D_2]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, which is the second column of A. Therefore,

$$D_1 = 1E_{11} + 0E_{12} + 1E_{21} + 0E_{22} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and}$$

$$D_2 = 0E_{11} + 1E_{12} + 0E_{21} + 1E_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

[**Alternate solution for parts (b) and (d)** via definitions of kernel and image. One can describe the basis for Kernel and Image by finding a spanning set for each using the transformation $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ a+2c & b+2d \end{bmatrix}$ as in Problem # 2.]

(d) (i) Let $\gamma = \{M_1, M_2, M_3, M_4\}$ (It must have 4 elements in it as the space $M_{2\times 2}$ is 4 dimensional). Then we know that the j-th column of the change of basis matrix $[\mathrm{I}]_\gamma^\beta = [M_j]_\beta$, that is, the coordinate vector of M_j with respect to the standard basis β.

Therefore, we get

$$\gamma = \{\underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix}1 & 0\\1 & -1\end{bmatrix}}_{M_3}, \underbrace{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}}_{M_4}\}$$

(ii) $[\mathrm{I}]_\beta^\gamma = ([\mathrm{I}]_\gamma^\beta)^{-1}$. (I skip the details here). Then

$$[\mathrm{I}]_\beta^\gamma = \begin{bmatrix}1 & -1 & -1 & 0\\-1/2 & 1 & 1 & 1/2\\1/2 & 0 & 0 & -1/2\\-1/2 & 0 & 1 & 1/2\end{bmatrix}$$

(iii) $[T]_\gamma^\beta = \underbrace{[\mathrm{I}]_\beta^\beta}_{=\mathrm{I}_4}[T]_\beta^\beta[\mathrm{I}]_\gamma^\beta = [T]_\beta^\beta[\mathrm{I}]_\gamma^\beta$.

Therefore,

$$[T]_\gamma^\beta = \begin{bmatrix}1 & 1 & 3 & 2\\2 & 3 & -2 & -1\\1 & 1 & 3 & 2\\2 & 3 & -2 & -1\end{bmatrix}.$$

For practice, verify that the columns of $[T]_\gamma^\beta$ are, indeed. the β-coordinate vectors of the images of $M_i \in \gamma$, $[T(M_i)]_\beta$, for each $i = 1, 2, 3, 4$

(iv) $[T]_\gamma^\gamma = [\mathrm{I}]_\beta^\gamma \underbrace{[T]_\beta^\beta[\mathrm{I}]_\gamma^\beta}_{[T]_\gamma^\beta \text{ by (iii)}}$ (which can be also observed as $[T]_\gamma^\gamma = Q^{-1}[T]_\beta^\beta Q$ where $Q = [\mathrm{I}]_\gamma^\beta$.)

Then we get

$$[T]_\gamma^\gamma = \begin{bmatrix}-2 & -3 & 2 & 1\\7/2 & 5 & -3/2 & -1/2\\-1/2 & -1 & 5/2 & 3/2\\3/2 & 2 & 1/2 & 1/2\end{bmatrix}$$

15. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(\begin{bmatrix}2\\1\end{bmatrix}) = \begin{bmatrix}4\\2\\-3\end{bmatrix} \quad \text{and} \quad T(\begin{bmatrix}2\\2\end{bmatrix}) = \begin{bmatrix}5\\0\\0\end{bmatrix}$$

Compute $T(\begin{bmatrix}6\\-1\end{bmatrix})$.

Answer. (Notice that T is given on a basis for $\mathbb{R}^2$.) We first need to write $\left[\begin{smallmatrix}6\\-1\end{smallmatrix}\right]$ as a linear combination of $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$. That is, solve $x\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right] + y\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right] = \left[\begin{smallmatrix}6\\-1\end{smallmatrix}\right]$, which gives the system $\left[\begin{array}{cc|c}2 & 2 & 6\\1 & 2 & -1\end{array}\right]$.

Row reduction (detailed omitted here) leads to $\left[\begin{array}{cc|c}1 & 0 & 7\\0 & 1 & -4\end{array}\right]$, i.e., $x = 7$, $y = -4$.

[A quick check shows $7\begin{bmatrix}2\\1\end{bmatrix} - 4\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}6\\-1\end{bmatrix}$.] Thus, by linearity of T we get

$T(\begin{bmatrix}6\\-1\end{bmatrix}) = T(7\begin{bmatrix}2\\1\end{bmatrix} - 4\begin{bmatrix}2\\2\end{bmatrix}) = 7\,T(2,1) - 4\,T(2,2) = 7\begin{bmatrix}4\\2\\-3\end{bmatrix} - 4\begin{bmatrix}5\\0\\0\end{bmatrix} = \begin{bmatrix}8\\14\\-21\end{bmatrix}$.