Math 271, Spring 2022

Extra Practice Problems for Exam

The problems and solutions below are gratefully borrowed, with minor modifications, from practice problems written by Rob Benedetto and Sema Gunturkun.

NOTE: These problems are not required and may not cover every single topic. They are simply for practice, to help you get ready for the first exam.

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1. Determine if each of following is **True** or **False**. If it is true, then give a short proof to justify your answer. If it is false, then either explain why clearly or give a precise counter example.

True **False** If $S = \{\vec{v}\}$ contains only one vector, then S must be always linearly independent.

True **False** If $S_1 \cap S_2 = \emptyset$ for two subsets S_1, S_2 of a vector space V, then $\text{Span}(S_1) \cap \text{Span}(S_2)$ must be the zero space $\{\vec{0}\}$.

True **False** For a given vector $\vec{v}$ in a vector space V. For any two distinct vectors $\vec{u}, \vec{w} \in \text{Span}(\{\vec{v}\})$, the subset $\{\vec{u}, \vec{w}\}$ is linearly dependent.

True **False** If a homogeneous system $\begin{matrix} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \end{matrix}$ has only the trivial solution, then

the inhomogeneous system $\begin{matrix} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{matrix}$ is consistent for every $(b_1, b_2, b_3) \in \mathbb{R}^3$.

True **False** Every homogeneous system of linear equations is consistent.

True **False** $\mathbb{Z}$ denotes the set of all integers in $\mathbb{R}$. $\mathbb{Z}$ is a subspace of $\mathbb{R}$.

2. Let $S = \{(0, \ 1, \ 0), (3, \ 17, \ -4), (0, \ 0, \ 1), (1, \ 0, \ 0)\} \subseteq \mathbb{R}^3$.

(a) Is S linearly independent? Why or why not?

(b) Does S span $\mathbb{R}^3$? Why or why not?

3. Let $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$. (Here "column vector notation" is used.)

Is T linearly independent? Why or why not?

4. Given subsets below in $P_2(\mathbb{R})$;

(I) $\{1+x, \ 2, \ 1-x+x^2\}$

(II) $\{1-x, \ 2, \ x-x^2, \ 2-2x+x^2\}$.

(III) $\{1, \ x+x^2, \ 2x+2x^2\}$.

(IV) $\{1-x, \ 1+x, \ x^2\}$

Determine which of them is a spanning set for $P_2(\mathbb{R})$.

5. Let $S = \{x+5, x^2-2, 3x^3-x+1, x^3+2x^2-2\}$.

(a) Is S linearly independent? Why or why not?

(b) Does S span $P_3(R)$? Why or why not?

6. Find the set of **all** solutions $(x, y, z, w) \in \mathbb{R}^4$ to the following system of equations.

$$\begin{aligned} x - 2y + 3z - 2w &= 2 \\ x + y \quad\quad + 4w &= 8 \\ 2x + y + z + 6w &= 14 \end{aligned}$$

8. Let $V = P_2(\mathbb{R})$, and let $W = \{p \in V \mid p(1) = 2p(-1)\} \subseteq V$.

(a) Prove that W is a subspace of V.

(b) Find a basis for W. [Don't forget to justify that your answer **is** a basis for W.]

9. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$. Prove that W is **not** a subspace of $\mathbb{R}^3$.

10. Let V be a vector space, and let $S \subseteq T \subseteq V$ be subsets.

(a) If T is linearly independent, prove that S is linearly independent.