1. Determine if each of following is True or False. If it is true, then give a short proof to justify your answer. If it is false, then either explain why clearly or give a precise counter example.

<table>
<thead>
<tr>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>True</strong></td>
<td>( S = { \vec{v} } ) contains only one vector, then ( S ) must be always linearly independent.</td>
</tr>
<tr>
<td><strong>False</strong></td>
<td>It is a false statement, it is not “always” linearly independent since it doesn’t say that ( \vec{v} ) is nonzero. If ( \vec{v} = \vec{0} ) then ( S ) is linearly dependent.</td>
</tr>
<tr>
<td><strong>True</strong></td>
<td>( S_1 \cap S_2 = \emptyset ) for two subsets ( S_1, S_2 ) of a vector space ( V ), then ( \text{Span}(S_1) \cap \text{Span}(S_2) ) must be the zero space ( { \vec{0} } ).</td>
</tr>
<tr>
<td><strong>False</strong></td>
<td>( \text{Span}(S_1) \cap \text{Span}(S_2) ) may or may not be the zero space. E.g. Consider ( \mathbb{R}^2 ), ( S_1 = { (1, 0) } ) and ( S_2 = { (-1, 0) } ), ( S_1 \cap S_2 = \emptyset ) however, ( \text{Span}(S_1) = \text{Span}(S_2) = x-axis ), so ( \text{Span}(S_1) \cap \text{Span}(S_2) = x-axis ), clearly not the zero space.</td>
</tr>
<tr>
<td><strong>True</strong></td>
<td>For a given vector ( \vec{v} ) in a vector space ( V ). For any two distinct vectors ( \vec{u}, \vec{w} \in \text{Span}({ \vec{v} }) ), the subset ( { \vec{u}, \vec{w} } ) is linearly dependent.</td>
</tr>
<tr>
<td><strong>False</strong></td>
<td>Proof. Since ( \vec{u}, \vec{w} \in \text{Span}({ \vec{v} }) ), we know ( \vec{u} = a\vec{v} ) and ( \vec{w} = b\vec{v} ) for some scalars ( a, b \in \mathbb{R} ). If either ( a ) or ( b ) is 0, then ( { \vec{u}, \vec{w} } ) is linearly dependent, since any set containing the zero vector is linearly dependent. On the other hand, if both ( a, b ) are nonzero, then ( \frac{1}{a} \vec{u} - \frac{1}{b} \vec{w} = \vec{v} - \vec{v} = \vec{0} ), which is a nontrivial linear dependence.</td>
</tr>
<tr>
<td><strong>True</strong></td>
<td>If a homogeneous system ( a_{21}x + a_{22}y + a_{23}z = 0 ) has only the trivial solution, then ( a_{31}x + a_{32}y + a_{33}z = 0 )</td>
</tr>
<tr>
<td><strong>False</strong></td>
<td>the inhomogeneous system ( a_{21}x + a_{22}y + a_{23}z = b_1 ) is consistent for every ( (b_1, b_2, b_3) \in \mathbb{R}^3 ).</td>
</tr>
<tr>
<td><strong>True</strong></td>
<td>Proof. A homogeneous linear systems with 3 equations in 3 variables has only the trivial solution if and only if its augmented matrix can be reduced to</td>
</tr>
</tbody>
</table>

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Then, for any \((b_1, b_2, b_3) \in \mathbb{R}^3\), the augmented matrix of the inhomogeneous system can be reduced to
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a_{22} & a_{23} & b_2 \\
a_{31} & a_{32} & a_{33} & b_3
\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 0 & b_1^* \\0 & 1 & 0 & b_2^* \\0 & 0 & 1 & b_3^*
\end{bmatrix}
\]
since its augmented matrix (no matter what \(b_i\) s are) never gets a row \([0 \ 0 \ 0 \ 1]\) in its reduced row-echelon form, the linear system is consistent for any \((b_1, b_2, b_3) \in \mathbb{R}^3\).

**True**   **False**  Every homogeneous system of linear equations is consistent.

Homogeneous systems always have the trivial solution, so they are always consistent.

**True**   **False**  \(\mathbb{Z}\) denotes the set of all integers in \(\mathbb{R}\). \(\mathbb{Z}\) is a subspace of \(\mathbb{R}\).

Take \(c = \frac{1}{2} \in \mathbb{R}\) as a scalar, and \(n = 3 \in \mathbb{Z}\), then \(cn = \frac{3}{2} \notin \mathbb{Z}\), that is, \(\mathbb{Z}\) is not closed under the scalar multiplication, thus \(\mathbb{Z}\) is not a subspace of \(\mathbb{R}\).
2. Let \( S = \{ (0, 1, 0), (3, 17, -4), (0, 0, 1), (1, 0, 0) \} \subseteq \mathbb{R}^3 \).

(a) Is \( S \) linearly independent? Why or why not?

(b) Does \( S \) span \( \mathbb{R}^3 \)? Why or why not?

**Answer.** (a) No, \# \( S \) = 4, and we discussed in class the fact that \( \mathbb{R}^n \) can never have a linearly independent subset with more than \( n \) vectors.

(b) Yes, it does because the standard basis of \( \mathbb{R}^3 \) is contained in \( S \), i.e. \( \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \} \subseteq S \), therefore \( \mathbb{R}^3 = \text{Span}(\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}) \subseteq \text{Span} S \subseteq \mathbb{R}^3 \), so we have \( \text{Span} S = \mathbb{R}^3 \).

3. Let \( T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right\} \subseteq \mathbb{R}^3 \). (Here “column vector notation” is used.)

Is \( T \) linearly independent? Why or why not?

**Answer.** No. Given \( x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). This equation gives a linear system whose augmented matrix is as follows, and its row reduction gives us

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 3 & -1 & 0 \\
1 & -2 & 4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 2 & -2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

which means that \( x_3 \) is a free variable (corresponds to the 3rd column, as it is a non-pivot column) so if we choose \( x_3 \) to be a nonzero value, then we get nontrivial relation between the vectors of \( T \).

4. Given subsets below in \( P_2(\mathbb{R}) \);

(I) \( \{ 1 + x, 2, 1 - x + x^2 \} \)

(II) \( \{ 1 - x, 2, x - x^2, 2 - 2x + x^2 \} \)

(III) \( \{ 1, x + x^2, 2x + 2x^2 \} \)

(IV) \( \{ 1 - x, 1 + x, x^2 \} \)

Determine which of them is a spanning set for \( P_2(\mathbb{R}) \).

**Answer.** Below I will use two approach to determine if they \( P_2(\mathbb{R}) \).

(I) \( S_1 = \{ 1 + x, 2, 1 - x + x^2 \} \) spans \( P_2(\mathbb{R}) \) because

- \( 1 \in \text{Span}(S_1) \), by multiplying constant polynomial 2 by scalar \( \frac{1}{2} \in \mathbb{R} \),
- then \( x \in \text{Span}(S_1) \), by the linear combination \( (1 + x) + (\text{a scalar } \frac{1}{2}) = x \in \text{Span}(S_1) \),

also \( (\text{a scalar } \frac{1}{2})^2 \)
• finally, \( x^2 \in \text{Span}(S_1) \), as we already got 1 and \( x \), then \( (1 - x + x^2) + x + (-1)1 = x^2 \text{Span}(S_1) \).

Thus \( \{1, x, x^2\} \subset \text{Span}(S_1) \) and we know that it is a basis for \( P_2(\mathbb{R}) \) so \( \text{Span}(\{1, x, x^2\}) = P_2(\mathbb{R}) \). Then \( P_2(\mathbb{R}) = \text{Span}(\{1, x, x^2\}) \subset \text{Span}(S_1) \subset P_2(\mathbb{R}) \). Therefore, \( \text{Span}(S_1) = P_2(\mathbb{R}) \).

(II) \( S_2 = \{1 - x, 2, x - x^2, 2 - 2x + x^2\} \) spans \( P_2(\mathbb{R}) \). Similar observation above, we can get \( \{1, x, x^2\} \subset \text{Span}(S_2) \).

Note that the above observation is simply answering the question “Are the elements of the basis \( \{1, x, x^2\} \) of \( P_2(\mathbb{R}) \) in \( \text{Span}(S_1) \)?” We are able to answer these questions quickly because \( S_1 \)'s elements are not so bad.

(III) \( S_3 = \{1, x + x^2, 2x + 2x^2\} \). Let’s use another (more standard method) technique to check; if \( S_3 \) spans \( P_2(\mathbb{R}) \), we should be able to write ANY polynomial \( p(x) = a + bx + cx^2 \) as a linear combination of elements of \( S_3 \). In other words, for any \( p(x) \) we should be able to find some scalars \( y_1, y_2, y_3 \in \mathbb{R} \) such that

\[
\begin{align*}
y_1(1) + y_2(x + x^2) + y_3(2x + 2x^2) &= a + bx + cx^2 \\
y_1(1) + (y_2 + 2y_3)x + (y_2 + 2y_3)x^2 &= a + bx + cx^2
\end{align*}
\]

Then we get a linear system (by matching the coefficients of the quadratic polynomials on each side)

\[
\begin{align*}
y_1 &= a \\
y_2 + 2y_3 &= b \\
y_2 + 2y_3 &= c
\end{align*}
\]

Easily we can see that this system has no solution whenever \( b \neq c \) (we don’t even need any Gauss-Jordan Elimination). So, for example, \( x + 3x^2 \notin \text{Span}(S_3) \), thus \( S_3 \) doesn’t span \( P_2(\mathbb{R}) \).

If you don’t enjoy working with arbitrary scalars \( a, b, c \), you can just check if \( x \) and \( x^2 \) are in the span; by checking the system when \( a = 0, b = 1, c = 0 \) and \( a = 0, b = 0, c = 1 \), respectively.

(IV) \( S_4 = \{1 - x, 1 + x, x^2\} \). As we did in case (I) and (II), notice that we already have \( x^2 \) in \( S_4 \subset \text{Span}(S_4) \). Then \( \frac{1}{2}(1 - x) + \frac{1}{2}(1 + x) = 1 \) so \( 1 \in \text{Span}(S_4) \). Finally, \( (1 + x) - 1 = x \in \text{Span}(S_4) \). Therefore, \( S_4 \) spans \( P_2(\mathbb{R}) \).

5. Let \( S = \{x + 5, x^2 - 2, 3x^3 - x + 1, x^3 + 2x^2 - 2\} \).

(a) Is \( S \) linearly independent? Why or why not?

(b) Does \( S \) span \( P_3(\mathbb{R}) \)? Why or why not?

Answer. Write \( S = \{f_1, f_2, f_3, f_4\} \). Consider \( a_1, a_2, a_3, a_4 \in \mathbb{R} \) with

\[
a_1f_1 + a_2f_2 + a_3f_3 + a_4f_4 = 0,
\]
(i.e., \((a_1, a_2, a_3, a_4)\) is a solution to the system, if we can find a nontrivial solution then it means \(S\) is linearly dependent.) Matching coefficients of the polynomial on left hand side with the zero polynomial on the right hand side gives us a linear system in unknowns \(a_i\); then we get

\[
\begin{pmatrix}
5 & -2 & 1 & -2 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 3 & 1 & 0
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
5 & -2 & 1 & -2 & 0 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 3 & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
0 & 2 & 6 & -2 & 0 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 3 & 1 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 3 & 1 & 0
\end{pmatrix}
\]

So \(a_4\) is free, and \(a_1 = a_3 = -a_4/3\), and \(a_2 = -2a_4\). Thus, we may choose, say, \(a_4 = 3\), so that \(a_1 = a_3 = -1\) and \(a_2 = -6\). So \(\text{No}; S\) is linearly dependent.

6. Find the set of all solutions \((x, y, z, w)\) \(\in \mathbb{R}^4\) to the following system of equations.

\[
\begin{align*}
x - 2y + 3z - 2w &= 2 \\
x + y + 4w &= 8 \\
2x + y + z + 6w &= 14
\end{align*}
\]

**Answer:** Row reduction gives

\[
\begin{align*}
R_1 \begin{bmatrix}
1 & -2 & 3 & -2 & 2
\end{bmatrix} &\rightarrow R_2 - R_1 \begin{bmatrix}
1 & -2 & 3 & -2 & 2
\end{bmatrix} \\
R_2 \begin{bmatrix}
1 & 1 & 0 & 4 & 8
\end{bmatrix} &\rightarrow R_3 - 2R_1 \begin{bmatrix}
0 & 3 & -3 & 6 & 6
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
R_3 \begin{bmatrix}
1 & -2 & 3 & -2 & 2
\end{bmatrix} &\rightarrow R_1 + 3R_3 \begin{bmatrix}
1 & 0 & 1/3 & 0
\end{bmatrix} \\
R_2 \begin{bmatrix}
0 & 1 & -1 & 2 & 2
\end{bmatrix} &\rightarrow R_2 + 3R_3 \begin{bmatrix}
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
\]

That is, \(x = 6 - z - 2w\), and \(y = 2 + z - 2w\), with \(z, w\) free. Thus, let \(w = t\) and \(z = s\) be free parameters, then solution set \(X\) is

\[
X = \{(6 - s - 2t, 2 + s - 2t, s, t) \mid s, t \in \mathbb{R}\} = \{(6, 2, s, t) + t(-2, -2, 0, 1) + s(-1, 1, 1, 0) \mid s, t \in \mathbb{R}\}
\]
8. Let $V = P_2(\mathbb{R})$, and let $W = \{ p \in V \mid p(1) = 2p(-1) \} \subseteq V$.

(a) Prove that $W$ is a subspace of $V$.

(b) Find a basis for $W$. [Don’t forget to justify that your answer is a basis for $W$.]

**Answer. (a) Proof.** We have $0 \in W$, since $0(1) = 0 = 2 \cdot 0 = 20(-1)$.

Given $p, q \in W$, we have

$$(p + q)(1) = p(1) + q(1) = 2p(-1) + 2q(-1) = 2(p(-1) + q(-1)) = 2(p + q)(-1),$$

so $p + q \in W$.

Finally, given $p \in W$ and $c \in \mathbb{R}$, we have

$$(cp)(1) = c(p(1)) = c(2p(-1)) = 2cp(-1),$$

so $ap \in W$. Thus, $W$ is a subspace of $V$. \(\square\)

(b) An element of $W$ is a polynomial $p(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$ and $a + b + c = p(1) = 2p(-1) = 2a - 2b + 2c$. That is, the condition on $a, b, c \in \mathbb{R}$ is that $c = 3b - a$. In other words,

$$W = \{ ax^2 + bx + (3b - a) : a, b \in \mathbb{R} \} = \{ a(x^2 - 1) + b(x + 3) : a, b \in \mathbb{R} \} = \text{Span}(S),$$

where $S = \{ x^2 - 1, x + 3 \}$. Moreover, $S$ is linearly independent, since neither of the two vectors is a scalar multiple of the other. (That’s because the $x^2$ term of $x + 3$ is zero, and the $x$ term of $x^2 - 1$ is zero.)

Thus, $S = \{ x^2 - 1, x + 3 \}$ is a basis for $W$.

9. Let $W = \{ (x, y, z) \in \mathbb{R}^3 \mid z \geq 0 \}$. Prove that $W$ is **not** a subspace of $\mathbb{R}^3$.

**Proof.** We have $(1, 1, 1) \in W$ and $-1 \in \mathbb{R}$, but

$$-1(1, 1, 1) = (-1, -1, -1) \notin W,$$

So $W$ is not closed under scalar multiplication and hence is not a subspace. \(\square\)

10. Let $V$ be a vector space, and let $S \subseteq T \subseteq V$ be subsets.

(a) If $T$ is linearly independent, prove that $S$ is linearly independent.

(b) If $\text{Span}(S) = V$, prove that $\text{Span}(T) = V$.

**Proof.** Given $\vec{x}_1, \ldots, \vec{x}_n \in S$ distinct and $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n = \vec{0}$, we have $\vec{x}_1, \ldots, \vec{x}_n \in T$. Since $T$ is linearly independent, we have $a_1 = a_2 = \cdots = a_n = 0$.

$\therefore$ $S$ linearly independent. \(\square\)

(b) **Proof.** Clearly $\text{Span}(T) \subseteq V$, since $T$ is a subset of $V$, and $V$ is a vector space.

For the reverse inclusion: given $\vec{v} \in V$, there exist $\vec{x}_1, \ldots, \vec{x}_n \in S$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that $\vec{v} = a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n$, since $\vec{v} \in V = \text{Span}(S)$.

Thus, $\vec{x}_1, \ldots, \vec{x}_n \in T$ (because $S \subseteq T$), and we get $\vec{v} \in \text{Span}(T)$. Hence, $V \subseteq \text{Span}(T)$.

$\therefore$ $V = \text{Span}(T)$ \(\square\)