Math 271, Spring 2022

Solutions to Extra Practice Problems for Exam 1

The problems and solutions below are gratefully borrowed, with minor modifications, from practice problems written by Rob Benedetto and Sema Gunturkun.

NOTE: These problems are not required and may not cover every single topic. They are simply for practice, to help you get ready for the first exam.

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1. Determine if each of following is **True** or **False**. If it is true, then give a short proof to justify your answer. If it is false, then either explain why clearly or give a precise counter example.

True $\boxed{\textbf{False}}$ If $S = \{\vec{v}\}$ contains only one vector, then S must be always linearly independent.

It is a false statement, it is not "always" linearly independent since it doesn't say that $\vec{v}$ is nonzero. If $\vec{v} = \vec{0}$ then S is linearly dependent.

True $\boxed{\textbf{False}}$ If $S_1 \cap S_2 = \emptyset$ for two subsets S_1, S_2 of a vector space V, then $\text{Span}(S_1) \cap \text{Span}(S_2)$ must be the zero space $\{\vec{0}\}$.

$\text{Span}(S_1) \cap \text{Span}(S_2)$ may or may not be the zero space. E.g. Consider $\mathbb{R}^2$, $S_1 = \{(1,0)\}$ and $S_2 = \{(-1,0)\}$, $S_1 \cap S_2 = \emptyset$ however, $\text{Span}(S_1) = \text{Span}(S_2) = x - axis$, so $\text{Span}(S_1) \cap \text{Span}(S_2) = x - axis$, clearly not the zero space.

$\boxed{\textbf{True}}$ **False** For a given vector $\vec{v}$ in a vector space V. For any two distinct vectors $\vec{u}, \vec{w} \in \text{Span}(\{\vec{v}\})$, the subset $\{\vec{u}, \vec{w}\}$ is linearly dependent.

Proof. Since $\vec{u}, \vec{w} \in \text{Span}(\{\vec{v}\})$, we know $\vec{u} = a\vec{v}$ and $\vec{w} = b\vec{v}$ for some scalars $a, b \in \mathbb{R}$. If either a or b is 0, then $\{\vec{u}, \vec{w}\}$ is linearly dependent, since any set containing the zero vector is linearly dependent. On the other hand, if both a, b are nonzero, then $\frac{1}{a}\vec{u} - \frac{1}{b}\vec{w} = \vec{v} - \vec{v} = \vec{0}$, which is a nontrivial linear dependence. □

$\boxed{\textbf{True}}$ **False** If a homogeneous system
$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= 0 \\ a_{21}x + a_{22}y + a_{23}z &= 0 \\ a_{31}x + a_{32}y + a_{33}z &= 0 \end{aligned}$$
has only the trivial solution, then the inhomogeneous system
$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$
is consistent for every $(b_1, b_2, b_3) \in \mathbb{R}^3$.

Proof. A homogeneous linear systems with 3 equations in 3 variables has only the trivial solution if and only if its augmented matrix can be reduced to

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Then, for any $(b_1, b_2, b_3) \in \mathbb{R}^3$, the augmented matrix of the inhomogeneous system can be reduced to

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1^* \\ 0 & 1 & 0 & b_2^* \\ 0 & 0 & 1 & b_3^* \end{array}\right]$$

since its augmented matrix (no matter what b_i s are) never gets a row $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$ in its reduced row-echelon form, the linear system is consistent for any $(b_1, b_2, b_3) \in \mathbb{R}^3$. $\square$

$\boxed{\textbf{True}}$ **False** Every homogeneous system of linear equations is consistent.

Homogeneous systems always have the trivial solution, so they are always consistent. $\square$

True $\boxed{\textbf{False}}$ $\mathbb{Z}$ denotes the set of all integers in $\mathbb{R}$. $\mathbb{Z}$ is a subspace of $\mathbb{R}$.

Take $c = \frac{1}{2} \in \mathbb{R}$ as a scalar, and $n = 3 \in \mathbb{Z}$, then $cn = \frac{3}{2} \notin \mathbb{Z}$, that is, $\mathbb{Z}$ is not closed under the scalar multiplication, thus $\mathbb{Z}$ is not a subspace of $\mathbb{R}$.

2. Let $S = \{(0,\ 1,\ 0), (3,\ 17,\ -4), (0,\ 0,\ 1), (1,\ 0,\ 0)\} \subseteq \mathbb{R}^3$.

(a) Is S linearly independent? Why or why not?

(b) Does S span $\mathbb{R}^3$? Why or why not?

Answer. (a) No, $\#S = 4$, and we discussed in class the fact that $\mathbb{R}^n$ can never have a linearly independent subset with more than n vectors.

(b) Yes, it does because the standard basis of $\mathbb{R}^3$ is contained in S, i.e. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \subset S$, therefore

$$\mathbb{R}^3 = \operatorname{Span}(\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}) \subseteq \operatorname{Span} S \subseteq \mathbb{R}^3,$$

so we have $\operatorname{Span} S = \mathbb{R}^3$.

3. Let $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$. (Here "column vector notation" is used.)

Is T linearly independent? Why or why not?

Answer. No. Given $x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This equation gives a linear system whose augmented matrix is as follows, and its row reduction gives us

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 3 & -1 & 0 \\ 1 & -2 & 4 & 0 \end{array}\right] \to \begin{array}{c} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$$

$$\to \begin{array}{c} \\ \frac{R_2}{2} \\ \frac{R_3}{3} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \to \begin{array}{c} R_1 - R_2 \\ \\ R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which means that x_3 is a free variable (corresponds to the 3rd column, as it is a non-pivot column) so if we choose x_3 to be a nonzero value, then we get nontrivial relation between the vectors of T.

4. Given subsets below in $P_2(\mathbb{R})$;

(I) $\{1 + x,\ 2,\ 1 - x + x^2\}$

(II) $\{1 - x,\ 2,\ x - x^2,\ 2 - 2x + x^2\}$.

(III) $\{1,\ x + x^2,\ 2x + 2x^2\}$.

(IV) $\{1 - x,\ 1 + x,\ x^2\}$

Determine which of them is a spanning set for $P_2(\mathbb{R})$.

Answer. Below I will use two approach to determine if they $P_2(\mathbb{R})$.

(I) $S_1 = \{1 + x,\ 2,\ 1 - x + x^2\}$ spans $P_2(\mathbb{R})$ because

- $1 \in \operatorname{Span}(S_1)$, by multiplying constant polynomial 2 by scalar $\frac{1}{2} \in \mathbb{R}$,
- then $x \in \operatorname{Span}(S_1)$, by the linear combination $(1 + x) + \underbrace{(-1)1}_{\text{also } (\frac{-1}{2})2} = x \in \operatorname{Span}(S_1))$,

- finally, $x^2 \in \text{Span}(S_1)$, as we already got 1 and x, then $(1-x+x^2)+x+(-1)1 = x^2\,\text{Span}(S_1)$.

Thus $\{1, x, x^2\} \subset \text{Span}(S_1)$ and we know that it is a basis for $P_2(\mathbb{R})$ so $\text{Span}(\{1,x,x^2\}) = P_2(\mathbb{R})$. Then $P_2(\mathbb{R}) = \text{Span}(\{1,x,x^2\}) \subseteq \text{Span}(S_1) \subseteq P_2(\mathbb{R})$. Therefore, $\text{Span}(S_1) = P_2(\mathbb{R})$.

(II) $S_2 = \{1-x,\ 2,\ x-x^2,\ 2-2x+x^2\}$ spans $P_2(\mathbb{R})$. Similar observation above, we can get $\{1, x, x^2\} \subset \text{Span}(S_2)$.

Note that the above observation is simply answering the question " Are the elements of the basis $\{1, x, x^2\}$ of $P_2(\mathbb{R})$ in $\text{Span}(S_1)$? " . We are able to answer these questions quickly because S_1's elements are not so bad.

(III) $S_3 = \{1,\ x+x^2,\ 2x+2x^2\}$. Let's use another(more standard method) technique to check ; if S_3 spans $P_2(\mathbb{R})$, we should be able to write ANY polynomial $p(x) = a+bx+cx^2$ as a linear combination of elements of S_3. In other words, for any $p(x)$ we should be able to find some scalars $y_1, y_2, y_3 \in \mathbb{R}$ such that

$$\begin{aligned} y_1(1) + y_2(x+x^2) + y_3(2x+2x^2) &= a+bx+cx^2 \\ y_1(1) + (y_2+2y_3)x + (y_2+2y_3)x^2 &= a+bx+cx^2 \end{aligned}$$

Then we get a linear system (by matching the coefficients of the quadratic polynomials on each side)

$$\begin{aligned} y_1 &= a \\ y_2 + 2y_3 &= b \\ y_2 + 2y_3 &= c \end{aligned}$$

Easily we can see that this system has no solution whenever $b \neq c$ (we don't even need any Gauss-Jordan Elimination). So, for example, $x+3x^2 \notin \text{Span}(S_3)$, thus S_3 doesn't span $P_2(\mathbb{R})$.

If you don't enjoy working with arbitrary scalars a, b, c, you can just check if x and x^2 are in the span; by checking the system when $a=0, b=1, c=0$ and $a=0, b=0, c=1$, respectively.

(IV) $S_4 = \{1-x,\ 1+x,\ x^2\}$ As we did in case (I) and (II), notice that we already have x^2 in $S_4 \subset \text{Span}(S_4)$. Then $\frac{1}{2}(1-x)+\frac{1}{2}(1+x) = 1$ so $1 \in \text{Span}(S_4)$. Finally, $(1+x)-1 = x \in \text{Span}(S_4)$. Therefore, S_4 spans $P_2(\mathbb{R})$.

5. Let $S = \{x+5, x^2-2, 3x^3-x+1, x^3+2x^2-2\}$.
   (a) Is S linearly independent? Why or why not?
   (b) Does S span $P_3(R)$? Why or why not?

Answer. Write $S = \{f_1, f_2, f_3, f_4\}$. Consider $a_1, a_2, a_3, a_4 \in \mathbb{R}$ with

$$a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 = 0,$$

(i.e., (a_1, a_2, a_3, a_4) is a solution to the system, if we can find a nontrivial solution then it means S is linearly dependent.) Matching coefficients of the polynomial on left hand side with the zero polynomial on the right hand side gives us a linear system in unknowns a_i; then we get

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \\ \mathrm{R}_4 \end{array} \left[\begin{array}{cccc|c} 5 & -2 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{array}\right] \to \begin{array}{c} \mathrm{R}_2 \\ \mathrm{R}_1 \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 5 & -2 & 1 & -2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \mathrm{R}_2 - 5\mathrm{R}_1 \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{array}\right] \to \begin{array}{c} \\ \frac{\mathrm{R}_2}{(-2)} \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ \mathrm{R}_3 - \mathrm{R}_2 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{array}\right] \to \begin{array}{c} \\ \\ \\ \mathrm{R}_4 - \mathrm{R}_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ \frac{\mathrm{R}_3}{3} \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{c} \mathrm{R}_1 + \mathrm{R}_3 \\ \mathrm{R}_2 + 3\mathrm{R}_3 \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So a_4 is free, and $a_1 = a_3 = -a_4/3$, and $a_2 = -2a_4$, Thus, we may choose, say, $a_4 = 3$, so that $a_1 = a_3 = -1$ and $a_2 = -6$. So **No**; S is linearly dependent.

6. Find the set of **all** solutions $(x, y, z, w) \in \mathbb{R}^4$ to the following system of equations.

$$\begin{aligned} x - 2y + 3z - 2w &= 2 \\ x + y \qquad + 4w &= 8 \\ 2x + y + z + 6w &= 14 \end{aligned}$$

Answer: Row reduction gives

$$\begin{array}{c} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array} \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 2 \\ 1 & 1 & 0 & 4 & 8 \\ 2 & 1 & 1 & 6 & 14 \end{array}\right] \to \begin{array}{c} \\ \mathrm{R}_2 - \mathrm{R}_1 \\ \mathrm{R}_3 - 2\mathrm{R}_1 \end{array} \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 2 \\ 0 & 3 & -3 & 6 & 6 \\ 0 & 5 & -5 & 10 & 10 \end{array}\right]$$

$$\to \begin{array}{c} \\ \frac{\mathrm{R}_2}{3} \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & -2 & 3 & -2 & 2 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 5 & -5 & 10 & 10 \end{array}\right] \to \begin{array}{c} \mathrm{R}_1 + 2\mathrm{R}_2 \\ \\ \mathrm{R}_3 - 5\mathrm{R}_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

That is, $x = 6 - z - 2w$, and $y = 2 + z - 2w$, with z, w free. Thus, let $w = t$ and $z = s$ be free parameters, then solution set X is

$$\begin{aligned} X &= \{(6 - s - 2t, 2 + s - 2t, s, t) \quad | \quad s, t \in \mathbb{R}\} \\ &= \{(6,\ 2,\ 0,\ 0) + t(-2,\ -2,\ 0,\ 1) + s(-1,\ 1,\ 1,\ 0) \quad | \quad s, t \in \mathbb{R}\} \end{aligned}$$

8. Let $V = P_2(\mathbb{R})$, and let $W = \{p \in V \mid p(1) = 2p(-1)\} \subseteq V$.

(a) Prove that W is a subspace of V.

(b) Find a basis for W. [Don't forget to justify that your answer **is** a basis for W.]

Answer. (a) Proof. We have $\vec{0} \in W$, since $\vec{0}(1) = 0 = 2 \cdot 0 = 2\vec{0}(-1)$.
Given $p, q \in W$, we have

$$(p+q)(1) = p(1) + q(1) = 2p(-1) + 2q(-1) = 2(p(-1) + q(-1)) = 2(p+q)(-1),$$

so $p + q \in W$.
Finally, given $p \in W$ and $c \in \mathbb{R}$, we have

$$(cp)(1) = c(p(1)) = c(2p(-1)) = 2cp(-1),$$

so $ap \in W$. Thus, W is a subspace of V. $\square$

(b) An element of W is a polynomial $p(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$ and $a + b + c = p(1) = 2p(-1) = 2a - 2b + 2c$. That is, the condition on $a, b, c \in \mathbb{R}$ is that $c = 3b - a$. In other words,

$$W = \{ax^2 + bx + (3b - a) : a, b \in \mathbb{R}\} = \{a(x^2 - 1) + b(x + 3) : a, b \in \mathbb{R}\} = \mathrm{Span}(S),$$

where $S = \{x^2 - 1, x + 3\}$. Moreover, S is linearly independent, since neither of the two vectors is a scalar multiple of the other. (That's because the x^2 term of $x + 3$ is zero, and the x term of $x^2 - 1$ is zero.)
Thus, $S = \{x^2 - 1, x + 3\}$ is a basis for W.

9. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$. Prove that W is **not** a subspace of $\mathbb{R}^3$.

Proof. We have $(1, 1, 1) \in W$ and $-1 \in \mathbb{R}$, but

$$-1(1, 1, 1) = (-1, -1, -1) \notin W,$$

So W is not closed under scalar multiplication and hence is not a subspace. $\square$

10. Let V be a vector space, and let $S \subseteq T \subseteq V$ be subsets.

(a) If T is linearly independent, prove that S is linearly independent.

(b) If $\mathrm{Span}(S) = V$, prove that $\mathrm{Span}(T) = V$.

(a) Proof. Given $\vec{x}_1, \ldots, \vec{x}_n \in S$ distinct and $a_1, \ldots, a_n \in \mathbb{R}$ such that $a_1\vec{x}_1 + \cdots a_n\vec{x}_n = \vec{0}$, we have $\vec{x}_1, \ldots, \vec{x}_n \in T$. Since T is linearly independent, we have $a_1 = a_2 = \cdots = a_n = 0$.
$\therefore$ S linearly independent. $\square$

(b) Proof. Clearly $\mathrm{Span}(T) \subseteq V$, since T is a subset of V, and V is a vector space.
For the reverse inclusion: given $\vec{v} \in V$, there exist $\vec{x}_1, \ldots, \vec{x}_n \in S$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that $\vec{v} = a_1\vec{x}_1 + \cdots a_n\vec{x}_n$, since $\vec{v} \in V = \mathrm{Span}(S)$.
Thus, $\vec{x}_1, \ldots, \vec{x}_n \in T$ (because $S \subseteq T$), and we get $\vec{v} \in \mathrm{Span}(T)$. Hence, $V \subseteq \mathrm{Span}(T)$.
$\therefore$ $V = \mathrm{Span}(T)$ $\square$