Textbook reading for this week:

- Chapter 1.1 (vector spaces) and 1.2 (subspaces).

Study items:

- Vector spaces are the most important objects in linear algebra. (This is why we do them right at the beginning so you have plenty of time to get used to them.) You should know by heart the eight properties that make up the definition of a vector space.
- Understand the phrases "for all" and "there exists" in the definition of a vector space. These phrases are extremely important in all of mathematics. What do these phrases tell you about the task of proving (or disproving!) that a given set and operations form a vector space?
- Notation: $\mathbb{R}^n$, $F(\mathbb{R})$, $P_n(\mathbb{R})$ for certain vector spaces; $\mathbf{0}, \vec{0}$, $\mathbf{x}, \vec{x}$ for vectors.
- Be able to prove facts that are true in any vector space (e.g. Proposition 1.1.6 in the text).
- Know, and be able to use, the definition of a subspace (of a vector space).
- How do you prove that a given subset is, or is not, a subspace?

Problems:

1. *(Damiano–Little 1.1.1)* Let $\vec{x} = (1, 3, 2), \vec{y} = (-2, 3, 4), \vec{z} = (-3, 0, 3)$ in $\mathbb{R}^3$.
   (a) Compute $3\vec{x}$.
   (b) Compute $4\vec{x} - \vec{y}$.
   (c) Compute $-\vec{x} + \vec{y} + 3\vec{z}$.

Note. You'll notice that I have used the notation $\vec{x}$ above, which is also what I will generally handwrite in class, whereas our text instead writes a boldface $\mathbf{x}$. Both notations are common, and either is fine if you are typing.

2. *(Damiano–Little 1.1.2)* Let $f = 3e^{3x}, g = 4e^{3x} + e^x, h = 2e^x - e^{3x}$ in $F(\mathbb{R})$.
   (a) Compute $5f$.
   (b) Compute $2f + 3g$.
   (c) Compute $-2f - g + 4h$.
3. *(Damiano–Little 1.1.3, axioms 4,6 only)* Prove that $\mathbb{R}^n$ satisfies axioms 4 and 6 from Definition (1.1.1).
4. *(Damiano–Little 1.1.4, axioms 3.8 only)* Prove that $P_n(\mathbb{R})$, with the operations given in Example (1.1.3), satisfies axioms 3 and 8 from Definition (1.1.1).
5. *(Damiano–Little 1.1.6(a,c))* In each of the following parts, decide if the set $\mathbb{R}^2$, with the given operations, is a vector space. If not, say which of the axioms fail to hold.
   (a) vector sum $(x_1, x_2) +' (y_1, y_2) = (x_1 + 2y_1, 3x_2 - y_2)$, and the usual scalar multiplication $c(x_1, x_2) = (cx_1, cx_2)$.
   (c) vector sum $(x_1, x_2) +' (y_1, y_2) = (0, x_1 + y_2)$, and the usual scalar multiplication.

6. *(Damiano–Little 1.1.7(c))* Suppose that the $F(\mathbb{R})$ (see definition on p. 6) is given the following two operations: the "sum" is defined by $f +' g = f \circ g$ (composition of functions), and the scalar multiplication is the usual operation. Decide whether or not the set $F(\mathbb{R})$, with these operations, is a vector space. If not, say which of the axioms fail to hold.

7. *(Damiano–Little 1.1.8)* Show that in any vector space V,

   (a) If $\vec{x}, \vec{y}, \vec{z} \in V$, then $\vec{x} + \vec{y} = \vec{x} + \vec{z}$ implies $\vec{y} = \vec{z}$.

   (b) If $\vec{x}, \vec{y} \in V$ and $a, b \in \mathbb{R}$, then $(a + b)(\vec{x} + \vec{y}) = a\vec{x} + b\vec{x} + a\vec{y} + b\vec{y}$.

8. Prove that in any vector space, $c\vec{0} = \vec{0}$ for all $c \in \mathbb{R}$.

9. *(Damiano–Little 1.2.1)* Show that for any vector space V, the set $\{\vec{0}\}$ is a subspace of V. (It is sometimes called the *trivial subspace*)

Note. The textbook has solutions to selected exercises in the back, including this one. You may use these to check your work. In this case, though, the solution in the book is incomplete, because it does not explain how we know that $c\vec{0} = \vec{0}$ for any $c \in \mathbb{R}$, which is why I have added the exercise above.

10. *(Damiano–Little 1.2.2(a,b))* Let $V_1 = \{f : \mathbb{R} \to \mathbb{R} \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}$. This is called the set of *even functions.*

    (a) Show that $\cos(x)$ and x^2 define functions in V_1.

    (b) Show that V_1 is a subspace of $F(\mathbb{R})$, using the same operations given in Example (1.2.1).

11. Let W denote the subset of $F(\mathbb{R})$ consisting of functions f that satisfy $f''(x) = -f(x)$. For example, W includes the functions $\sin(x)$ and $\cos(x)$. Prove that W is a subspace of $F(\mathbb{R})$.

12. Suppose that W is a subspace of $F(\mathbb{R})$ (not necessarily the same W as in the previous problem). Suppose that $f, g \in W$ are two elements of W that satisfy $f(0) = 1, f'(0) = 0, g(0) = 0$ and $g'(0) = 1$. Prove that there exists an element $h \in W$ such that $h(0) = 3$ and $h'(0) = 7$.

Extra practice (not to hand in)

- *(Damiano–Little 1.1.3, all properties)*
- *(Damiano–Little 1.1.4, all properties)*
- *(Damiano–Little 1.1.6(b))*
- *(Damiano–Little 1.1.7(a,b))*
- *(Damiano–Little 1.1.10)*
- *(Damiano–Little 1.1.11)*
- *(Damiano–Little 1.2.2)*
- *(Damiano–Little 1.2.3)*
- *(Damiano–Little 1.2.8)*
- *(Damiano–Little 1.2.9)*