Textbook reading for this week:

- §2.3 (kernel and image)
- Suggested: begin reading §2.4 (applications of rank-nullity)

Study items:

- How do you calculate the matrix associated to a linear transformation with respect to given ordered bases?
- How do you calculate the linear transformation associated to a matrix with respect to given ordered bases?
- How do you use coordinate representations of objects to make calculations for a linear transformation?
- H ow do youcompute a basis for the kernel and image of a specific linear transformation?

Problems:

Note. Before solving the problems below, read the definition of the "identity transformation" in Example 2.1.4(a) (p. 65).

1. *(Damiano–Little 2.2.6)* (matrix representations for some transformations related to the identity)
2. *(Damiano–Little 2.2.9(a))* (matrix representation of the identity using different bases)
3. *(Damiano–Little 2.2.12(a))* (transposition is a linear transformation)
4. Let $T : P_3(\mathbb{R}) \to \mathbb{R}^4$ be the function given by

$$T(p(x)) = (p(0), p'(0), p''(0), p'''(0)).$$

   You may take my word for it that T is linear. Calculate the matrix of T with respect to the standard bases of $P_3(\mathbb{R})$ and $\mathbb{R}^4$.

5. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the function given by

$$T(p(x)) = xp(x) - p'(x).$$

   (a) Prove that T is a linear transformation.
   (b) Calculate the matrix $[T]_\alpha^\beta$, where $\alpha = \{1, x, x^2\}$ and $\beta = \{1, x, x^2, x^3\}$, are the standard bases of $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$, respectively.
   (c) Calculate the coordinate vectors $[2x^2 - x - 3]_\alpha$ and $[T(2x^2 - x - 3)]_\beta$.
   (d) Do matrix multiplication to verify that the equation

$$[T(2x^2 - x - 3)]_\beta = [T]_\alpha^\beta [2x^2 - x - 3]_\alpha$$

   is true.

6. *(Damiano–Little 2.3.3(b,c))* (basis for kernel and image of 2×3 and 3×6 matrix)

7. *(Damiano–Little 2.3.7(a))* (find a transformation with given vectors in kernel and image)

8. *(Damiano–Little 2.3.13)* (symmetric and skew-symmetric matrices as a kernel and image)

9. Let $\alpha = \{(1,0,1),(0,1,1),(1,1,0)\}$ and $\beta = \{(1,1),(-1,1)\}$ be ordered bases for $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be a linear transformation whose matrix with respect to α and β is
$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 4 \\ -1 & -3 & 2 \end{bmatrix}.$$
   (a) Work out a formula for $T((x_1, x_2, x_3))$.
   (b) Find the matrix of T with respect to the standard bases for $\mathbf{R}^3$ and $\mathbf{R}^2$.

10. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by
$$T((x,y)) = (2x - y, x + 3y, -y).$$
   (a) Write down the matrix of T with respect to the standard bases $\alpha = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbf{R}^2$, and $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbf{R}^3$.
   (b) Let $\alpha' = \{\mathbf{e}_2, \mathbf{e}_1\}$ be the standard basis of $\mathbf{R}^2$, but in the *opposite order*. Calculate the matrix $[T]_{\alpha'}^\beta$.
   (c) Let $\beta' = \{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2\}$ be the standard basis of $\mathbf{R}^3$, but *with the last two vectors swapped.* Calculate the matrix $[T]_\alpha^{\beta'}$.
   (d) Now let V, W be any finite-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Let α be an ordered basis of V, and let β be an ordered basis of W. Based on your answers to parts (a)-(c), write down a general rule for what happens to the matrix $[T]_\alpha^\beta$ when you (i) swap two of the vectors in the basis α, or (ii) swap two of the vectors in the basis β.

   (You do not need to prove that your rule is correct.)