Supplement to problem set 11
(Concerns material from §6.2 and the discussion in class of least-squares)

1. Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Find the vector $\vec{x}$ such that $\|A\vec{x} - \vec{b}\|$ is as small as possible.

2. Let V be an inner product space, $S = \{\vec{w}_1, \cdots, \vec{w}_n\}$ an orthogonal set of vectors in V, and let W be the span of S. Let $\vec{b}$ be any vector in V. Suppose that $\vec{w} \in W$ is chosen such that $\vec{b} - \vec{w}$ is orthogonal to all vectors in W.

   (a) Prove that for any other vector $\vec{w'} \in W$,
   $$\|\vec{b} - \vec{w'}\|^2 = \|\vec{b} - \vec{w}\|^2 + \|\vec{w} - \vec{w'}\|^2.$$

   (b) Deduce from (a) that $\vec{w}$ the vector in W that is closest to $\vec{b}$ (in the sense that $\|\vec{b} - \vec{w}\|$ is minimized).

   (c) Show that $\vec{w}$ may be computed explicitly as follows.
   $$\vec{w} = \sum_{i=1}^{n} \frac{\langle \vec{b}, \vec{w}_i \rangle}{\langle \vec{w}_i, \vec{w}_i \rangle} \vec{w}_i.$$
   (It suffices to show that the vector defined by this formula satisfies the property that $\vec{w} - \vec{b}$ is orthogonal to all of W.)

   The following problem will involve calculating some integrals that would be messy to do by hand. You may use a computer system of your choice to do these computations (e.g. it is fairly easy to compute them from a web browser using Wolfram Alpha).

3. Let V be the vector space of continuous functions on the interval $[-\pi, \pi]$, equipped with the following inner product.
   $$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$
   The set $S = \{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x)\}$ is an orthogonal set in V (you do not need to prove this). Let W be the span of S.

   (a) Let $f(x) = |x|$. Using problem 2, find the function $g(x) \in W$ that minimizes $\int_{-\pi}^{\pi}(f(x) - g(x))^2 dx$. Sketch the graphs of both $f(x)$ and $g(x)$ (use a computer or calculator; e.g. Wolfram Alpha can do this fairly easily).

   (b) Let $f(x) = x$. Using problem 2, find the function $g(x) \in W$ that minimizes $\int_{-\pi}^{\pi}(f(x) - g(x))^2 dx$. Sketch the graphs of both $f(x)$ and $g(x)$.

   The functions found in this problem are examples of Fourier approximations, which are used in compression algorithms such as JPEG (images) and MP3 (audio). The virtue of these approximations is that they closely resemble the original data, but require relatively little information (in the problem above, a function $g(x) \in W$ is determined by only 7 numbers, since $\dim W = 7$.).