Textbook problems from DeFranza and Gagliardi:

- §2.3: 6, 8, 36, 42
- §6.1: 14, 18, 23, 32
- §3.2: 18, 20, 24, 26, 30, 44, 50

Supplemental problems:
Note: You may wish to use Mathematica, or other software, for the computations in some of the problems below. If you do, clearly indicate which computations you are using software for, or print out and include the Mathematica input/output with your submission.

1. A square matrix A is called *orthogonal* if $A^tA = I$ (in other words: if it is invertible, and its inverse is equal to its transpose). Prove that if A is orthogonal, then $\det A$ is either 1 or -1.

   Solution: Since $\det(A^t) = \det A$, it follows that if A is orthogonal, then

$$\begin{aligned} \det(A^tA) &= \det I \\ \det(A^t)\det A &= 1 \\ \det(A)^2 &= 1 \end{aligned}$$

   Hence $\det A$ must be either 1 or -1 (these are the only two numbers that square to 1).

2. Find the linear combination of $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ that is closest to $\begin{pmatrix}5\\3\\7\end{pmatrix}$.

   Solution: The best linear combination will be $c_1\begin{pmatrix}1\\0\\-1\end{pmatrix}+c_2\begin{pmatrix}1\\-1\\0\end{pmatrix}$, where c_1, c_2 are solutions to the normal equation, which is, in this case:

$$\begin{aligned} \begin{pmatrix} \begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} & \begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} \\ \begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} & \begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} \end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} &= \begin{pmatrix} \begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}5\\3\\7\end{pmatrix} \\ \begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\3\\7\end{pmatrix} \end{pmatrix} \\ \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} &= \begin{pmatrix}-2\\2\end{pmatrix} \end{aligned}$$

   Solving this equation (either by row-reduction or inverting the 2×2 matrix) gives the (unique) solution

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}$$

   Hence the closest linear combination is $-2\begin{pmatrix}1\\0\\-1\end{pmatrix} + 2\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\-2\\2\end{pmatrix}$. (The minimize distance is $5\sqrt{3}$).

3. Define a matrix A and vector $\vec{b}$ as follows.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 8 \\ 4 \\ 12 \\ 0 \end{pmatrix}$$

(a) Verify that the linear system $A\vec{x} = \vec{b}$ is inconsistent.

Solution: To solve the linear system $A\vec{x} = \vec{b}$, we may reduce the augment matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 1 & 1 & 0 & 4 \\ 0 & 1 & 1 & 12 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

After performing (in order) the operations `R2 -= R1, R3 -= R2, R4 -= R3`, this reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 16 \\ 0 & 0 & 0 & -15 \end{array}\right).$$

Since the last row encodes the equation $0 = -15$, the linear system is inconsistent.

(b) Find the "least-squares" solution, i.e. the vector $\vec{x}$ which minimizes $\|A\vec{x} - \vec{b}\|$.

Solution: We solve the normal equation:

$$\begin{aligned} A^t A\vec{x} &= A^t\vec{b} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \\ 12 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \vec{x} &= \begin{pmatrix} 12 \\ 16 \\ 12 \end{pmatrix} \end{aligned}$$

Solving this equation (e.g. by row-reduction) gives $\vec{x} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$, which therefore minimizes $\|A\vec{x} - \vec{b}\|$ (the minimum distance is 8).

4. Consider the following four points in the plane. This problem will demonstrate a couple ways that we could find a "line of best fit" for these four points. Part (a) is the usual method. The purpose of this exercise is to see how choosing a different "objective" (function to be minimized) can product different "lines of best fit" (essentially because it depends on what "best" means, which may be different in different applications).

$$(x_1, y_1) = (1, 1) \qquad (x_2, y_2) = (3, 2) \qquad (x_3, y_3) = (1, 6) \qquad (x_4, y_4) = (3, 7)$$

(a) Suppose that we wish to find the coefficients c_1, c_2 that minimize the sum $\sum_{i=1}^{4}(c_1\, x_i + c_2 - y_i)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|$. Then find the optimal coefficients c_1, c_2. Sketch the four points and the line $y = c_1 x + c_2$.

Solution: Define, for convenience, the following vectors:

$$\vec{x} = \begin{pmatrix}1\\3\\1\\3\end{pmatrix} \qquad \vec{y} = \begin{pmatrix}1\\2\\6\\7\end{pmatrix} \qquad \vec{1} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

(I will refer to these in parts (b) and (c) as well).

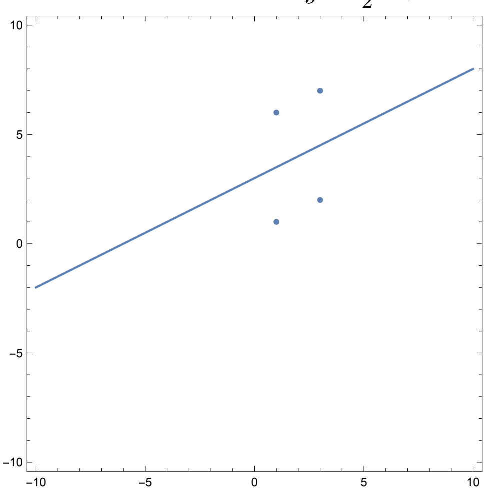
The sum being minimized in part (a) is equal to $\|c_1\vec{x} + c_2\vec{1} - \vec{y}\|^2$. So we may take $\vec{v}_1 = \vec{x},\ \vec{v}_2 = \vec{1},\ \vec{b} = \vec{y}$. The best choice of c_1, c_2 is found by solving the normal equation:

$$\begin{aligned}
\begin{pmatrix}\vec{v}_1\cdot\vec{v}_1 & \vec{v}_1\cdot\vec{v}_2\\ \vec{v}_2\cdot\vec{v}_1 & \vec{v}_2\cdot\vec{v}_2\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} &= \begin{pmatrix}\vec{v}_1\cdot\vec{b}\\ \vec{v}_2\cdot\vec{b}\end{pmatrix}\\
\begin{pmatrix}\vec{x}\cdot\vec{x} & \vec{x}\cdot\vec{1}\\ \vec{1}\cdot\vec{x} & \vec{1}\cdot\vec{1}\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} &= \begin{pmatrix}\vec{x}\cdot\vec{y}\\ \vec{1}\cdot\vec{y}\end{pmatrix}\\
\begin{pmatrix}20 & 8\\ 8 & 4\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} &= \begin{pmatrix}34\\16\end{pmatrix}
\end{aligned}$$

Solving this equation (by row-reducing or inverting the matrix) gives

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}1/2\\3\end{pmatrix}.$$

So the line of best fit is $y = \frac{1}{2}x + 3$. Here is a plot of that line with the four data points.



(b) Suppose that we now want coefficients c_1, c_2 that minimize $\sum_{i=1}^{4}(c_1 y_i + c_2 - x_i)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|$. Then find the optimal coefficients c_1, c_2 and sketch the four points and the line $x = c_1 y + c_2$.

Solution: The expression to be optimized is equal to $\|c_1\vec{y}+c_2\vec{1}-\vec{x}\|^2$. So this time we may take $\vec{v}_1=\vec{y}, \vec{v}_2=\vec{1}$, and $\vec{b}=\vec{x}$. The best choice of c_1, c_2 can be obtained from the normal equation:

$$\begin{pmatrix} \vec{v}_1\cdot\vec{v}_1 & \vec{v}_1\cdot\vec{v}_2 \\ \vec{v}_2\cdot\vec{v}_1 & \vec{v}_2\cdot\vec{v}_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{v}_1\cdot\vec{b} \\ \vec{v}_2\cdot\vec{b} \end{pmatrix}$$

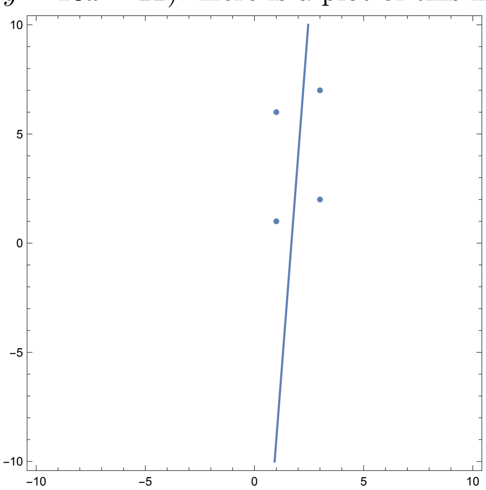
$$\begin{pmatrix} \vec{y}\cdot\vec{y} & \vec{y}\cdot\vec{1} \\ \vec{1}\cdot\vec{y} & \vec{1}\cdot\vec{1} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{y}\cdot\vec{x} \\ \vec{1}\cdot\vec{x} \end{pmatrix}$$

$$\begin{pmatrix} 90 & 16 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 34 \\ 8 \end{pmatrix}$$

Solving this by row-reduction or matrix inversion gives

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/13 \\ 22/13 \end{pmatrix}$$

So the line of best first (for our new cost function) is $x=\frac{1}{13}y+\frac{22}{13}$ (or, equivalently, $y=13x-22$). Here is a plot of this line with the four points.



Note that this is a much steeper line than as found in part (a). This is because it is optimizing the horizontal distance to the points, rather than the vertical distance. It turns out to be better to choose a steep line rather than a shallow line for this purpose.

(c) A third way to specify a line is using an equation of the form $c_1x+c_2y=1$. Suppose that now we wish to find c_1, c_2 minimizing $\sum_{i=1}^{4}(c_1x_i+c_2y_i-1)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1+c_2\vec{v}_2-\vec{b}\|$. Then find the optimal coefficients c_1, c_2 and sketch the four points with the line $c_1x+c_2y=1$.

Solution: The quantity to be optimized is equal to $\|c_1\vec{x}+c_2\vec{y}-\vec{1}\|$. So we may take $\vec{v}_1=\vec{x}, \vec{v}_2=\vec{y}$ and $\vec{b}=\vec{1}$. The best choice of c_1, c_2 can be found using the normal equation:

$$\begin{pmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \cdot \vec{b} \\ \vec{v}_2 \cdot \vec{b} \end{pmatrix}$$

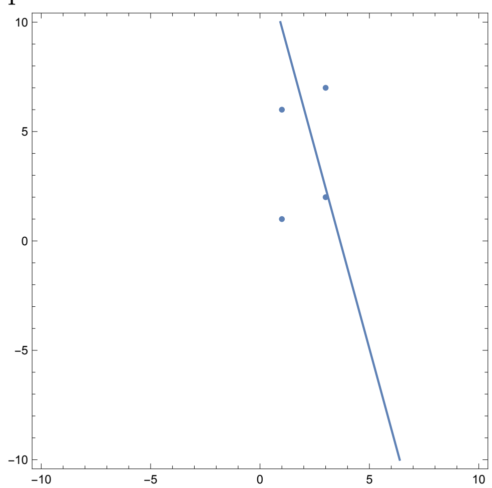
$$\begin{pmatrix} \vec{x} \cdot \vec{x} & \vec{x} \cdot \vec{y} \\ \vec{y} \cdot \vec{x} & \vec{y} \cdot \vec{x} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{x} \cdot \vec{1} \\ \vec{y} \cdot \vec{1} \end{pmatrix}$$

$$\begin{pmatrix} 20 & 34 \\ 34 & 90 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$$

Solving this by row-reduction of matrix inversion gives

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 44/161 \\ 12/161 \end{pmatrix}$$

So the line of best fit (for this cost function) is $\frac{44}{161}x + \frac{12}{161}y = 1$ (other equivalent forms: $44x + 12y = 161$, or $y = -\frac{44}{12}x + \frac{161}{12}$). Here is a plot of this line with the four data points.



(Same "important notes" and "submission instructions" apply as before; they will be omitted from now to reduce clutter.)