Textbook problems from DeFranza and Gagliardi:

- §6.2: 31
- §6.3: 14, 18, 23, 36 (Note: for the Gram-Schmidt problems, refer either to last week's lab or to the discussion in §6.3 for the details of the procedure).

Supplemental problems: The following problems both concern approximation in inner product spaces, as described in Wednesday's and Friday's classes. In both problems, feel free to use software to compute difficult integrals, solve complicated linear systems, or to make plots (see the notes after problem 2 for a suggestion).

1. (Approximation of functions by polynomials) Consider the function $f(x) = \sin(2\pi x)$ on the interval $[0, 1]$. This function can be regarded as a vector in the vector space $\mathcal{C}[0, 1]$ (this is our notation from class; the book uses the notation $\mathcal{C}^{(0)}([0, 1])$).

   (a) Find the degree-3 polynomial $p(x)$ which approximates $f(x)$ as well as possible, in the sense of minimizing the "total squared error"

   $$\int_0^1 (p(x) - f(x))^2 \, dx.$$

   Note that you may view $p(x)$ as a linear combination of $\{1, x, x^2, x^3\}$, which can play the role of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ in the statement of the "Main Theorem" in Wednesday's and Friday's classes.

   (b) Make a plot of both the function $f(x)$ and the polynomial $p(x)$ on the interval $[0, 1]$ to verify that $p(x)$ provides a good approximation.

   Remark: Although it is not necessary for this problem, in applications one often finds these polynomial approximations by first using Gram-Schmidt to replace $\{1, x, x^2, x^3\}$ by an *orthogonal* (or orthonormal) set, and then using the simplified approximation method discussed in Friday's class. You are of course free to follow this route if you wish, but I suspect that it is likely to require more work than using the method we discussed on Wednesday.

2. (Fourier approximations) Let V be the vector space of continuous functions on the interval $[-\pi, \pi]$, equipped with the following inner product.

   $$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

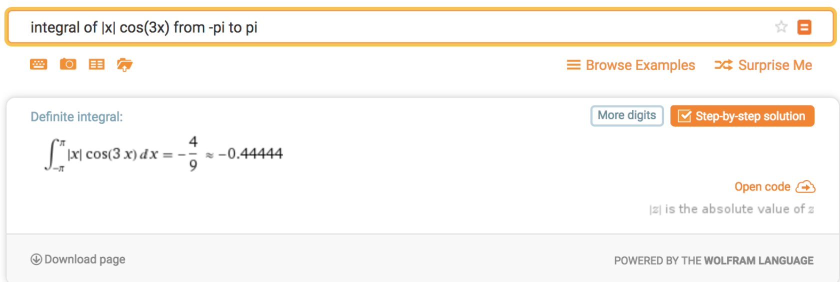
   The set $S = \{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x)\}$ is an orthogonal set in V, as discussed in class (you may take this as given).

   (a) Let $f(x) = |x|$. Find the linear combination $g(x)$ of S that minimizes $\int_{-\pi}^{\pi}(f(x) - g(x))^2 dx$. Sketch the graphs of both $f(x)$ and $g(x)$ (use a computer or calculator; e.g. Wolfram Alpha can do this fairly easily).

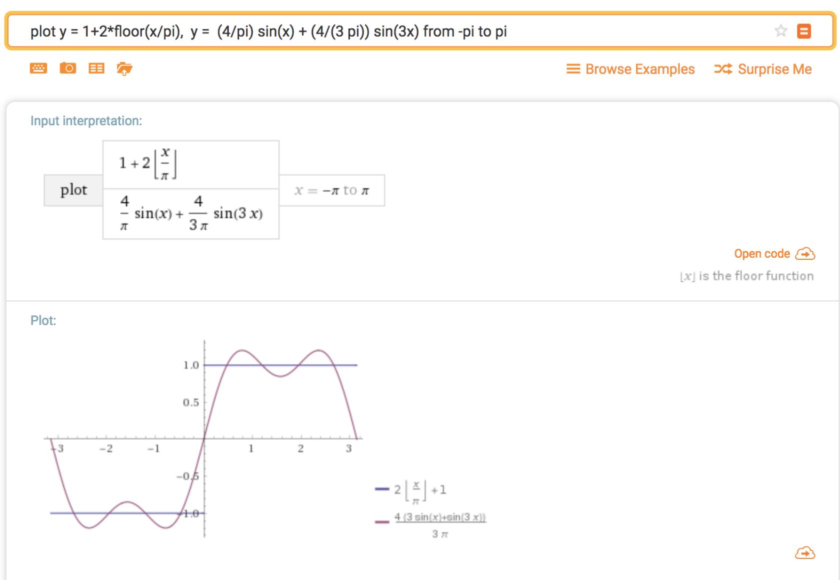
   (b) Let $f(x) = x$. Find the linear combination $g(x)$ of S that minimizes $\int_{-\pi}^{\pi}(f(x) - g(x))^2 dx$. Sketch the graphs of both $f(x)$ and $g(x)$.

   (c) From the coefficients you find in part (b), try to guess the linear combination that will best approximate $f(x) = x$ if we also include frequencies up to 7, i.e. expand S to $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \cdots, \sin(7x), \cos(7x)\}$. Plot your answer to see if it looks like a good approximation.

Note: Although the necessary integrals are tractable to compute by hand, I suggest that you save time by using a computer. A useful resource for this, accessible for any web browser, is Wolfram Alpha. The screenshot below shows that wording you can use to computer on of the integrals needed in part (a) of problem 2, for example.



Wolfram alpha is also a quick and easy way to plot the functions you find. The following screenshot shows the wording that can be used to plot the example from class on Friday (approximating the square wave).



Of course, you can also use Mathematica itself, if you are using a computer with it installed.

(Same "important notes" and "submission instructions" apply as before; omitted to save space.)