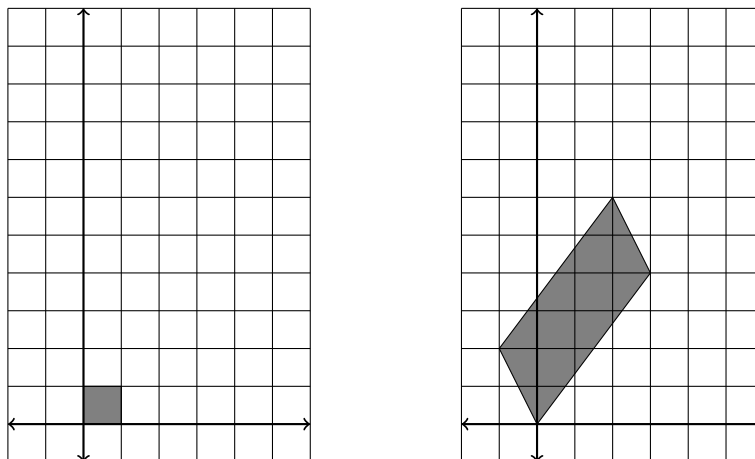


1. Consider a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that has the following effect on the unit square.



- (a) [2 points] Find the matrix representation of T in the standard basis.
(There are two possible answers based on how you interpret the picture. You need only give one.)
- (b) [2 points] Determine $R(T)$ and $N(T)$ (no explanation is necessary for this part).
- (c) [3 points] Find the matrix representation, in the standard basis, of the inverse transformation T^{-1} .
- (d) [3 points] Determine a point $\vec{x} \in \mathbb{R}^2$ that is sent to $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ by this transformation (i.e.

$$T(\vec{x}) = \begin{pmatrix} 5 \\ 10 \end{pmatrix}.$$

2. Let A be the following 3×5 matrix.

$$A = \begin{pmatrix} 2 & -6 & -9 & -11 & -8 \\ 2 & -6 & -6 & -8 & -4 \\ 1 & -3 & -3 & -4 & 0 \end{pmatrix}$$

The reduced row echelon form of A is as follows (you do not need to verify this yourself).

$$\begin{pmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- (a) [3 points] Give a basis for $N(A)$.
- (b) [3 points] Give a basis for $R(A)$.
- (c) [3 points] Let A be the same matrix above, and let \vec{b} be any vector in \mathbb{R}^3 . Explain why the matrix equation $A\vec{x} = \vec{b}$ is consistent, regardless of the choice of \vec{b} . How many free variables occur in the general solution of $A\vec{x} = \vec{b}$?
3. [9 points] Let

$$A = \begin{pmatrix} 2 & -2 & -3 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}.$$

Determine the inverse matrix A^{-1} .

4. Suppose that $T : V \rightarrow W$ is a linear transformation of vector spaces, and $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a set of three vectors in V .
- (a) [5 points] Suppose $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is a linearly independent set in W . Prove that S is a linearly independent set in V .
- (b) [5 points] Prove that if S is a linearly independent set in V and T is one-to-one, then $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is a linearly independent set in W .
5. [9 points] Denote by $\vec{u}, \vec{v}, \vec{b}$ the following three vectors in \mathbb{R}^4 .

$$\vec{u} = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ -11 \\ -11 \\ 11 \end{pmatrix}$$

Determine the linear combination \vec{w} of $\{\vec{u}, \vec{v}\}$ that is closest to \vec{b} (that is, the linear combination that minimizes $\|\vec{w} - \vec{b}\|$).

6. Let V be an inner product space. Suppose that W is a subspace of V with basis $B = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.
- (a) [2 points] What is the dimension of W ?
- (b) [4 points] Suppose that \vec{b} is a vector in V , and \vec{u} is a vector in W such that

$$\vec{w}_1 \perp (\vec{b} - \vec{u}), \quad \vec{w}_2 \perp (\vec{b} - \vec{u}), \quad \text{and} \quad \vec{w}_3 \perp (\vec{b} - \vec{u}).$$

Prove that $\vec{b} - \vec{u}$ is orthogonal to every vector in W .

- (c) [4 points] Let \vec{b}, \vec{u} be as in part (b). Prove that if \vec{v} is any other vector in W , then

$$\|\vec{b} - \vec{v}\|^2 = \|\vec{b} - \vec{u}\|^2 + \|\vec{u} - \vec{v}\|^2.$$

7. Define a map $T : \mathcal{P}_2 \rightarrow \mathbb{R}^4$ by the formula

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{pmatrix}.$$

T is a linear transformation (you do not need to prove this).

- (a) [4 points] Let $B = \{(x-1)(x-2), x(x-2), x(x-1)\}$. This is a basis of \mathcal{P}_2 (you do not need to prove this). Let S denote the standard basis of \mathbb{R}^4 . Determine the matrix representation $[T]_B^S$ of T with respect to the basis B of \mathcal{P}_2 and the standard basis S of \mathbb{R}^4 .
- (b) [3 points] Let A be the matrix $[T]_B^S$ obtained in the previous part. Prove that $\vec{b} \in R(T)$ if and only if the matrix equation $A\vec{x} = \vec{b}$ is consistent.
- (c) [3 points] Suppose that we are given four constants a, b, c, d , and wish to find a polynomial $p(x) \in \mathcal{P}_2$ whose graph passes through the four points $(0, a), (1, b), (2, c), (3, d)$. For which values of a, b, c, d is this possible? Express your answer as a linear equation in a, b, c , and d .

Hint: interpret this as asking when a linear system is consistent, using the previous part.

8. [9 points] Let A be an $n \times n$ matrix, and λ a scalar. Define as usual the following set (called the *eigenspace* in class).

$$V_\lambda = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}$$

Prove that V_λ is a *subspace* of \mathbb{R}^n .

9. In a certain town, 8000 customers subscribe to internet service from company A . A competitor, company B , enters the market and begins to draw customers from company A .

Suppose that every year, 10% of company A 's customers change their service to company B , and 30% of company B 's customers change their service to company A .

For example:

- In the first year, 800 customers (10% of 8000) change service from A and B , after which company A has 7200 customers and company B has 800 customers.
- In the second year, 720 customers (10% of 7200) switch from A to B , and 240 customers (30% of 800) switch from B to A . So after two years, company A has $7200 - 720 + 240 = 6720$ customers and company B has $800 + 720 - 240 = 1280$ customers.

- (a) [2 points] Find a 2×2 matrix M encoding the change in number of customers of each company from one year to the next. More precisely: if a, b denote the number of customers of companies A and B (respectively) in a given year, and a', b' denote the number of customers in the following year, the matrix M should satisfy

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M \begin{pmatrix} a \\ b \end{pmatrix}.$$

You can check your answer by verifying that

$$\begin{pmatrix} 7200 \\ 800 \end{pmatrix} = M \begin{pmatrix} 8000 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6720 \\ 1280 \end{pmatrix} = M^2 \begin{pmatrix} 8000 \\ 0 \end{pmatrix}.$$

- (b) [6 points] Find the eigenvalues of M , and an eigenvector for each eigenvalue.
- (c) [3 points] Express $\begin{pmatrix} 8000 \\ 0 \end{pmatrix}$ as a linear combination of the eigenvectors you found in part (b).
- (d) [3 points] Find a formula for $M^n \begin{pmatrix} 8000 \\ 0 \end{pmatrix}$, and use your formula to evaluate $\lim_{n \rightarrow \infty} M^n \begin{pmatrix} 8000 \\ 0 \end{pmatrix}$.