1. [9 points] **Short answer questions.** No explanations are necessary.

(a) State the dimension of each of the following vector spaces.

- $\mathbb{R}^5$ 5
- $M_{2\times 3}$ 6
- $\mathcal{P}_3$ 4

(b) Find the angle between the vectors $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

dot product: $0+2-2=0$

$\boxed{90^\circ}$ $(\pi/2)$

(c) For each of the following subsets of $\mathbb{R}^2$, determine whether or not it is a subspace of $\mathbb{R}^2$. You do not need to prove your answer; simply state "yes" or "no."

- $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : x+5y=0\right\}$ yes
- $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : y=x^2\right\}$ no
- $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : 2x+3y=1\right\}$ no
- $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : y=7x\right\}$ yes

2. [9 points] Consider the two bases $B = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $[I]_B^{B'}$.

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ \underset{-5}{5} & \underset{-5/3}{2} & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 1/3 & -1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \end{array}\right)$$

so $\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]_{B'} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$

Also, $\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{B'} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by inspection (it matches the second vector of B').

Hence $[I]_B^{B'} = \boxed{\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}}$

<u>alt. solution</u>

Observe that

$[I]_B^S = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ & $[I]_{B'}^S = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ (bundle B & B' as columns of a matrix).

hence

$$[I]_B^{B'} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3\cdot 2 - 1\cdot 5} \cdot \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{1} \cdot \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

3. [15 points] Consider the following three vectors in $\mathbb{R}^4$.

$$\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

Denote by W the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.

(a) Find a basis of W.

$$\begin{pmatrix} 1 & -2 & 0 \\ -1 & 2 & 3 \\ 1 & -2 & 0 \\ -1 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

pivots in columns 1 & 3 $\Rightarrow$ $\boxed{\{\vec{u}, \vec{w}\}}$ is a basis of W.

(b) What is the dimension of W?

$\boxed{2}$ (the basis has 2 elements).

(Parts (c) and (d) on reverse side)

(c) Find an *orthonormal* basis for W. (Recall that a basis is called *orthonormal* if any two vectors in the basis are orthogonal, and each vector has norm equal to 1.)

First, ~~the~~ replace $\vec{u}, \vec{w}$ by orthogonal vectors:

$$\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\vec{w} - \text{proj}_{\vec{u}}(\vec{w}) = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} - \frac{\vec{u}\cdot\vec{w}}{\vec{u}\cdot\vec{u}}\cdot\vec{u} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} - \frac{-4}{4}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Then normalize:

$$\frac{1}{\sqrt{1+1+1+1}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\frac{1}{\sqrt{1+4+1+0}}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{pmatrix}$$

$$\boxed{\left\{ \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{pmatrix} \right\}}$$

(d) What element of W is closest to the vector $\vec{b} = \begin{pmatrix} 12 \\ 0 \\ 0 \\ 0 \end{pmatrix}$? (In other words, which element $\vec{x}$ of W minimizes $\|\vec{x} - \vec{b}\|$?)

To minimize $\|c_1\vec{u} + c_2\vec{w} - \vec{b}\|$, we can solve the normal eq'n:

$$\begin{pmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{w} \\ \vec{w}\cdot\vec{u} & \vec{w}\cdot\vec{w} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{u}\cdot\vec{b} \\ \vec{w}\cdot\vec{b} \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ -4 & 10 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 4 & -4 & 12 \\ -4 & 10 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 6 & 12 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

So the closest point is $5\vec{u} + 2\vec{w} = 5\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 1 \\ 5 \\ -3 \end{pmatrix}}$

Another method: add the projections of $\vec{b}$ onto the ortho. vectors from (c).

4. [9 points] Suppose that $\{\vec{u}, \vec{v}\}$ is a basis for a vector space V. Prove that $\{3\vec{u}+2\vec{v},\ \vec{u}+\vec{v}\}$ is also a basis for V.

Linear independence

Suppose $c_1(3\vec{u}+2\vec{v}) + c_2(\vec{u}+\vec{v}) = \vec{0}$.

Then $(3c_1+c_2)\vec{u} + (2c_1+c_2)\vec{v} = 0$.

Since $\vec{u}, \vec{v}$ are LI, it follows that

$3c_1+c_2 = 0$ & $2c_1+c_2=0$,

ie. $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Reducing, $\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$,

so the only possible values of c_1, c_2 are $\underline{c_1=0}$, $\underline{c_2=0}$.

Hence $\{3\vec{u}+2v, \vec{u}+\vec{v}\}$ is LI.

Span

Since $\{3\vec{u}+2\vec{v}, \vec{u}+\vec{v}\}$ is LI, its span is 2-dimensional.

Now, V itself is 2-dim'l (it has a basis of two elements),

so the span of $\{3\vec{u}+2v, \vec{u}+\vec{v}\}$ must be all of V.

Hence $\boxed{\{3\vec{u}+2\vec{v}, \vec{u}+\vec{v}\}}$ is a basis for V.

Note Another way to check that the set spans: observe that

$\vec{u} = 1\cdot(3u+2\vec{v}) - 2(\vec{u}+\vec{v})$

& $\vec{v} = -1(3\vec{u}+2\vec{v}) + 3(\vec{u}+\vec{v})$

Hence $\{\vec{u},\vec{v}\} \subseteq \text{span}\{3\vec{u}+2\vec{v}, \vec{u}+\vec{v}\}$,

and therefore all of V lies in this span, since $\{\vec{u},\vec{v}\}$ spans V.

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(a) $\|x\|^2 = \langle x, x\rangle = \int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = 1/3$

$\Rightarrow \|x\| = \sqrt{\frac{1}{3}} = \boxed{\frac{1}{\sqrt{3}}}$ (or $\frac{\sqrt{3}}{3}$).

(b) $\langle x, x^2+1\rangle = \int_0^1 x(x^2+1)\, dx = \left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_0^1$

$= \frac{1}{4} + \frac{1}{2} = \boxed{3/4}$

(c) $\mathrm{proj}_x(x^2+1) = \frac{\langle x, x^2+1\rangle}{\langle x, x\rangle} \cdot x = \frac{3/4}{1/3} \cdot x$

$= \boxed{\frac{9}{4}x}$

(d) The ~~mult~~ norm $\|c\cdot x - (x^2+1)\| = \sqrt{\int_0^1 (cx - (x^2+1))^2 dx}$ is minimized by setting $c = \frac{9}{4}$.

In other words, $\frac{9}{4}x$ is the multiple of x closest to (x^2+1) (in the sense of the norm).

6. [9 points] Consider the following three vectors.

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

(a) Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

row-reduce:

$$\begin{bmatrix}1&1&1\\1&2&1\\1&3&2\end{bmatrix} \longrightarrow \begin{bmatrix}1&1&1\\0&1&0\\0&2&1\end{bmatrix}$$

$$\longrightarrow \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$$

$$\longrightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Pivot in each column $\Rightarrow$ sol'ns to $\begin{bmatrix}1&1&1\\1&2&1\\1&3&2\end{bmatrix}\vec{x} = \vec{0}$ have no free variables $\Rightarrow$ columns are lin. indep.

(b) Find the unique scalars c_1, c_2, c_3 such that the vector

$$\vec{v} = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

is equal to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

same row ops., but w/ the aug. matrix:

$$\left[\begin{array}{ccc|c}1&1&1&2\\1&2&1&1\\1&3&2&3\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&1&1&2\\0&1&0&-1\\0&2&1&1\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c}1&0&1&3\\0&1&0&-1\\0&0&1&3\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-1\\0&0&1&3\end{array}\right].$$

$\Rightarrow \boxed{c_1 = 0, \quad c_2 = -1, \quad c_3 = 3}$ ie. $\boxed{[\vec{v}]_B = \begin{pmatrix}0\\-1\\3\end{pmatrix}.}$