Note: this problem set will cover two weeks of material, and will therefore be longer than usual. This is because there will not be an assignment due on the day of the midterm. **This set is now complete.**

Study guide

- (§6.2) Understand the proof of the "Pythagorean theorem for inner product spaces."
- (online notes) understand the inner product space version of the least squares problem and the normal equation.
- (online notes) be able to find the linear combination of a list of functions that is closest (in the least-squares sense) to a given function on a particular interval.
- (online notes) know the special case: for an orthogonal set of vectors in an IPS, least-squares is solved by a sum of projections.
  Midterm 2 material ends here.
- (§4.1) Know the definition of a "linear transformation," and their basic properties.
- (§4.1) Be comfortable with the notation $T: V \to W$.
- (§4.2) Given a matrix A, know the definitions of $N(A)$ and $R(A)$, and how to compute bases for both of them (both are reformulations of algorithms studied earlier in the course).
- (§4.1) Understand why linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ is always represetented by an $m \times n$ matrix (and what "represented by" means in this context).
- (§4.2) Know the "rank-nullity theorem." (Theorem 5 in the textbook)

Textbook problems

(No textbook problems about material to be covered on midterm 2)

- §4.1: 26, 30, 39, 42
- §4.2: 22, 32, 40

Supplemental problems:

1. Suppose that $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthogonal set in an inner product space V. Prove that for any three scalars $a, b, c \in \mathbb{R}$,

$$\|a\vec{u} + b\vec{v} + c\vec{w}\|^2 = a^2\|\vec{u}\|^2 + b^2\|\vec{v}\|^2 + c^2\|\vec{w}\|^2.$$

Hint: apply the Pythagorean theorem for inner product spaces twice.

The following problems concern approximation in inner product spaces, as described in this week's classes (see also some typed notes on the website). In both problems, **feel free to use software to compute difficult integrals, solve complicated linear systems, or to make plots** (see the notes after problem 2 for a suggestion).

2. (Approximation of functions by polynomials) Consider the function $f(x) = \sin(2\pi x)$ on the interval $[0, 1]$. This function can be regarded as a vector in the vector space $\mathcal{C}[0, 1]$ (this is our notation from class; the book uses the notation $\mathcal{C}^{(0)}([0, 1])$).

   (a) Find the degree-3 polynomial $p(x)$ which approximates $f(x)$ as well as possible, in the sense of minimizing the "total squared error"

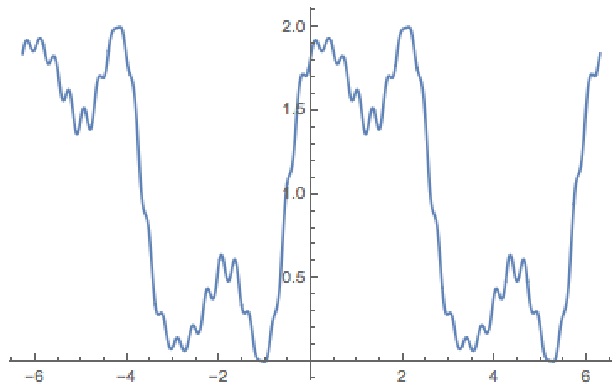
$$\int_0^1 (p(x) - f(x))^2 \, dx.$$

Note that you may view $p(x)$ as a linear combination of $\{1, x, x^2, x^3\}$, which can play the role of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ in our discussion of least-squares.

(b) Make a plot of both the function $f(x)$ and the polynomial $p(x)$ on the interval $[0, 1]$ to verify that $p(x)$ provides a good approximation.

Remark: Although it is not necessary for this problem, in applications one often finds these polynomial approximations by first using Gram-Schmidt to replace $\{1, x, x^2, x^3\}$ by an *orthogonal* (or orthonormal) set, and then using the simplified approximation method discussed in Friday's class.

3. You measure a signal in a laboratory, that appears to be 2π-periodic, with the following graph (shown on the interval $[-2\pi, 2\pi]$).



Call the function $f(x)$. In order to find a good approximation of this function, you compute the following integrals.

$$\int_{-\pi}^{\pi} f(x)\ dx = 6.283$$

$$\int_{-\pi}^{\pi} f(x)\sin x\ dx = 2.398 \qquad \int_{-\pi}^{\pi} f(x)\cos x\ dx = 1.379$$

$$\int_{-\pi}^{\pi} f(x)\sin(2x)\ dx = 0 \qquad \int_{-\pi}^{\pi} f(x)\cos(2x)\ dx = 0$$

$$\int_{-\pi}^{\pi} f(x)\sin(3x)\ dx = 0.360 \qquad \int_{-\pi}^{\pi} f(x)\cos(3x)\ dx = 1.379$$

From this information, find a linear combination $h(x)$ of the set

$$S = \{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x)\}$$

that minimizes the objective function $\int_{-\pi}^{\pi}(h(x) - f(x))^2\ dx$. Graph your function $h(x)$ (e.g. using Wolfram Alpha) and verify that it appears to give a decent approximation of the graph (make a sketch of the graph you obtain). You may use the fact (mentioned in class without proof) that S is an orthogonal set in $\mathcal{C}[-\pi, \pi]$, with the usual inner product.

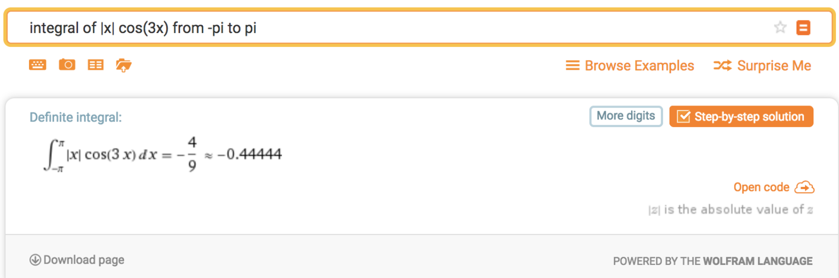
4. Let $S = \{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x)\}$ be as in the previous problem.

(a) Let $f(x) = |x|$. Find the linear combination $g(x)$ of S that minimizes $\int_{-\pi}^{\pi}(f(x) - g(x))^2 dx$. Sketch the graphs of both $f(x)$ and $g(x)$ (use a computer or calculator, e.g. use Wolfram Alpha). Remember that you may use software to compute any integrals.

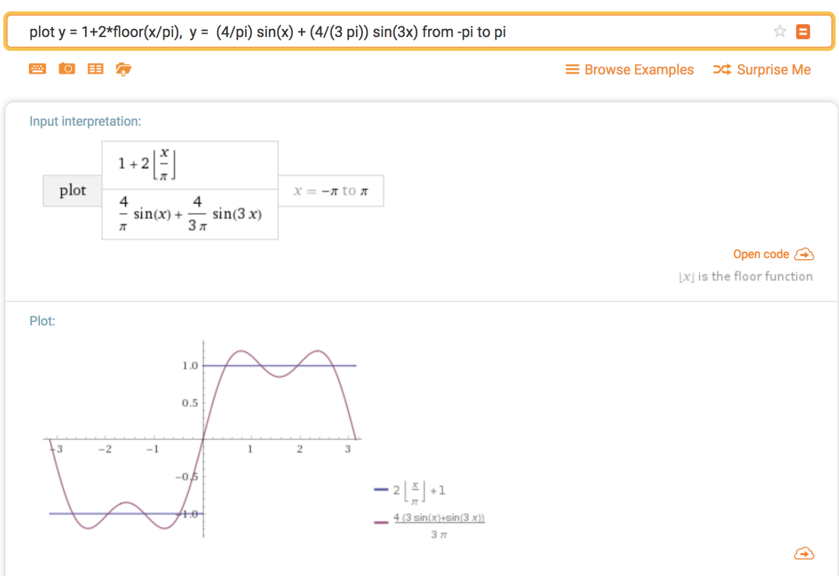
(b) Let $f(x) = x$. Find the linear combination $g(x)$ of S that minimizes $\int_{-\pi}^{\pi}(f(x) - g(x))^2 dx$. Sketch the graphs of both $f(x)$ and $g(x)$.

(c) From the coefficients you find in part (b), try to guess the linear combination that will best approximate $f(x) = x$ if we also include frequencies up to 7, i.e. expand S to $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \cdots, \sin(7x), \cos(7x)\}$. Plot your answer to see if it looks like a good approximation.

Note: Although the necessary integrals are tractable to compute by hand, I suggest that you save time by using a computer. A useful resource for this, accessible for any web browser, is Wolfram Alpha. The screenshot below shows that wording you can use to compute one of the integrals needed in part (a) of problem 4, for example.



Wolfram alpha is also a quick and easy way to plot the functions you find. The following screenshot shows the wording that can be used to plot a pair of functions (not from the examples above).



Of course, you can also use Mathematica itself, if you are using a computer with it installed.