Amherst College
Department of Mathematics and Statistics

MATH 272 FINAL EXAM FALL 2017

NAME: Solutions

Read This First!

- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. You may use the backs of pages for additional work space.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	12	9	9	9	9	9	12	9	9	9	96
Score:											

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1. [12 points] For each choice of a matrix A and vector $\vec{b}$ below, determine whether or not the linear system $A\vec{x} = \vec{b}$ is consistent. If it is consistent, find a solution. If it is inconsistent, find a "least-squares solution" (a vector $\vec{x}$ such that $\|A\vec{x} - \vec{b}\|$ is as small as possible).

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 2 & 4 & 5 \\ 1 & 3 & 9 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 4 & 4 \\ 0 & 3 & 9 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\boxed{\text{consistent; } \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}$$

(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & -1 \\ -1 & -2 & -3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\boxed{\text{consistent; } \vec{x} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}}$$

(Part (c) on the reverse side)

(c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{array}\right) \qquad \boxed{\text{inconsistent}}$$

so we solve the "normal equation":

$$A^T A \vec{x} = A^T \vec{b}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -2 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 3 \\ 3 & 6 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3 & 6 & 3 \\ 2 & 3 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 3 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right)$$

so $\boxed{\vec{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}}$ is the least-squares solution.

2. [9 points] Suppose that A is an $n \times n$ matrix and $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$. Show that if $A\vec{u} = A\vec{v}$ and $\vec{u} \neq \vec{v}$, then A is not invertible.

Sol'n 1 (by contradiction)

Suppose that $\vec{u} \neq \vec{v}$ and $A\vec{u} = A\vec{v}$.

Suppose, for contradiction, that A is invertible.

Then

$$A\vec{u} = A\vec{v}$$
$$\Rightarrow A^{-1}A\vec{u} = A^{-1}Av$$
$$\Rightarrow \vec{u} = \vec{v} \quad \text{(contradiction since } \vec{u} \neq \vec{v}\text{)}.$$

So A is not invertible after all.

Sol'n 2

If $A\vec{u} = A\vec{v}$ & $\vec{u} \neq \vec{v}$, then $A(\vec{u} - \vec{v}) = 0$.
Since $\vec{u} - \vec{v} \neq \vec{0}$, this means that $\vec{u} - \vec{v}$ is a nonzero element of the nullspace of A. But invertible matrices must have trivial nullspace, so A cannot be invertible.

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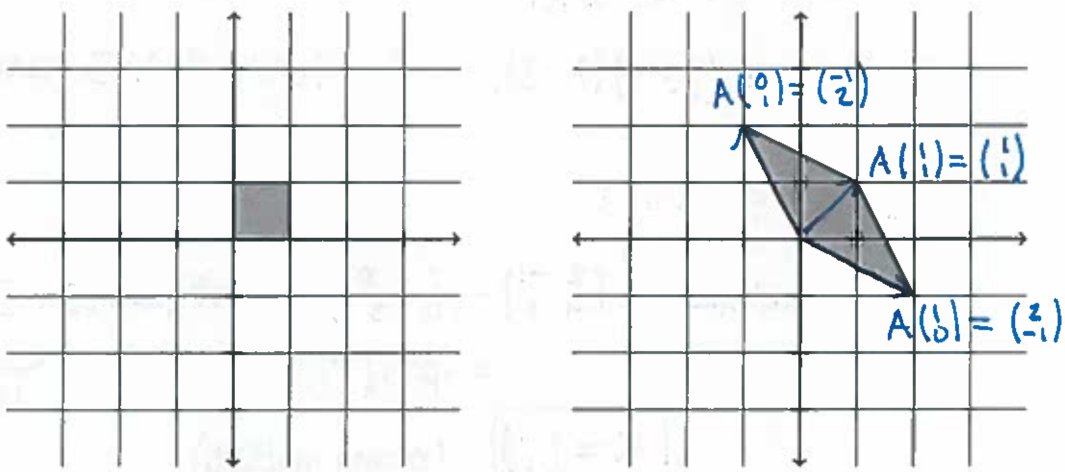
3. [9 points] Suppose that A is a 2×2 matrix, and that A represents (in the standard basis) a graphics operation that transforms the unit square as shown. The figure on the left shows the original unit square, while the figure on the right shows its image after transformation.



(a) Determine the matrix A.

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

& $A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$

so $\boxed{A = \begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix}}$

note: $A = \begin{pmatrix}-1 & 2\\2 & -1\end{pmatrix}$ is also a valid answer (the answers to (c) & (d) are different for this choice).

(b) Find a nonzero vector $\vec{v}$ such that $A\vec{v} = \vec{v}$.

$A\vec{v} = \vec{v} \iff (A-I)\vec{v} = 0$

& nullspace of $\begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix} - \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$

is $\operatorname{span}\begin{pmatrix}1\\1\end{pmatrix}$ (row-reduces to $\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}$).

$\boxed{\vec{v} = \begin{pmatrix}1\\1\end{pmatrix}}$ (or any multiple).

(Parts (c) and (d) on the reverse side)

(c) Find a nonzero vector $\vec{w}$ such that $A\vec{w}$ is a scalar multiple of $\vec{w}$, but is not equal to $\vec{w}$.

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - (-1)^2 = \lambda^2 - 4\lambda + 3$$

$$= (\lambda-1)(\lambda-3). \quad \text{So } \lambda = 1 \ \& \ \lambda = 3 \text{ are the eigenvalues.}$$

eigenvector for $\lambda = 3$:

$$\text{nullspace}\left(\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right) = \text{nullspace}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \underbrace{=}_{\text{(row-reducing)}} \text{nullspace}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \text{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

$$\boxed{\vec{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}} \quad \text{(or any multiple)}$$

(d) Let $B = \{\vec{v}, \vec{w}\}$, where $\vec{v}, \vec{w}$ are the vectors found in parts (b) and (c). What is the matrix representation of this graphics operation in the basis B?

$$A\vec{v} = \vec{v} = 1\cdot\vec{v} + 0\cdot\vec{w}, \quad \text{so} \quad [A\vec{v}]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$A\vec{w} = 3\vec{w} = 0\cdot\vec{v} + 3\cdot\vec{w}, \quad \text{so} \quad [A\vec{w}]_B = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

$$[A]_B = \begin{pmatrix} | & | \\ [A\vec{v}]_B & [A\vec{w}]_B \\ | & | \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}$$

4. [9 points] Let $\vec{u} = \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$. Find a basis of $\mathbb{R}^3$ that contains both $\vec{u}$ and $\vec{v}$.

we can row-reduce $\begin{pmatrix} | & | & | & | & | \\ \vec{u} & \vec{v} & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ | & | & | & | & | \end{pmatrix}$ and take the vectors corresponding to pivot columns.

$$\begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 \\ -8 & 4 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -4 & -8 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & -1/4 \\ 0 & 1 & 2 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & 1/4 \end{pmatrix} \quad \text{(RREF)}$$

so use columns 1, 2, & 4:

$$\boxed{\left\{ \begin{pmatrix} -1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.}$$

(many other answers are possible).

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5. [9 points] Denote by $\mathcal{P}_3$ the vector space of polynomials in one variable of degree at most 3. Define a transformation $T: \mathcal{P}_3 \to \mathcal{P}_3$ by $T(p(x)) = \frac{d}{dx}p(x)$ for all $p(x) \in \mathcal{P}_3$. For example,

$$T(x^3 + 4x^2) = 3x^2 + 8x.$$

(a) Show that T is a *linear* transformation.

For any two $p(x), q(x) \in \mathcal{P}_3$ and $c \in \mathbb{R}$,

$$\begin{aligned} T(p(x) + c \cdot q(x)) &= \tfrac{d}{dx}(p(x) + c \cdot q(x)) \\ &= p'(x) + c \cdot q'(x) \\ &= T(p) + c \cdot T(q) \checkmark \end{aligned}$$

so T is linear.

(b) Denote by $S = \{1, x, x^2, x^3\}$ the standard basis of $\mathcal{P}_3$. Find $[T]_S$, the matrix representation of T in the standard basis.

the columns of $[T]_S$ are found as follows:

$$[T(1)]_S = [0]_S = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x)]_S = [1]_S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x^2)]_S = [2x]_S = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x^3)]_S = [3x^2]_S = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Hence

$$\boxed{[T]_S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

(Parts (c) and (d) on the reverse side)

(c) What is the dimension of the null space of T?

RREF ~~to~~ of $[T]_S$ is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so nullspace is given (in S-coordinates) by span$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,

ie. span(1), all constant polynomials.

spanned by one nonzero vector

$\Rightarrow$ $\boxed{\dim \text{nullspace}(T) = 1}$.

(d) What is the dimension of the range of T?

sol'n 1 By rank-nullity,

$$\dim R(T) = \dim P_3 - \dim \text{nullspace}(T)$$
$$= 4 - 1$$
$$= \boxed{3}$$

sol'n 2 using the RREF of T, we see that $T(x), T(x^2), T(x^3)$ are a basis of $R(T)$, so $R(T)$ is 3-dimensional.

6. [9 points] Suppose that A is a square matrix, and n is a positive integer such that $A^n = \mathbf{0}$ (the all-0's matrix).

(a) Prove that the only eigenvalue of A is $\lambda = 0$.

Suppose λ is an eigenvalue.
Then $\exists \vec{v} \neq \vec{0}$ s.t. $A\vec{v} = \lambda\vec{v}$.
It follows that:

$$\begin{aligned} & A^n\vec{v} = \lambda^n\vec{v} \\ \Rightarrow\ & 0\vec{v} = \lambda^n\vec{v} \\ \Rightarrow\ & \vec{0} = \lambda^n \vec{v} \\ \Rightarrow\ & \lambda^n = 0 \quad (\text{since } \vec{v} \neq \vec{0}). \\ \Rightarrow\ & \underline{\lambda = 0}. \end{aligned}$$

So $\lambda = 0$ is the only possible eigenvalue.

(b) Prove that if A is nonzero, then A is *not* diagonalizable.

↯ Suppose A is diagonalizable and nonzero.

Then $A = PDP^{-1}$ where D is diagonal,
say $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$. ($m$ = # rows of A)

Then

$$\begin{aligned} & A^n = P\begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_m^n \end{pmatrix}P^{-1} \\ \Rightarrow\ & 0 = P\begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_m^n \end{pmatrix}P^{-1} \quad (\text{since } A^n = 0) \\ \Rightarrow\ & P^{-1}0P = \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_m^n \end{pmatrix} \\ \Rightarrow\ & 0 = \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_m^n \end{pmatrix}, \text{ so all } \lambda_1, \dots, \lambda_m \text{ are } 0. \\ \Rightarrow\ & D = 0. \end{aligned}$$

Or, more briefly: note that the entries of D must be eigenvalues of A, so by (a) they're all 0.

But this means that $A = P \cdot 0 P^{-1} = 0$. ↯.

So if $A \neq 0$, then it's not diagonalizable.

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7. [12 points] Consider the matrix $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) What are the eigenvalues of A?

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(3-\lambda)$$

$$\boxed{\lambda = 1, 2, 3}$$

(b) Find an eigenvector for each eigenvalue that you found in part (a).

$\underline{\lambda=1}$ $\begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{\text{(row-ops)}} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{pmatrix} \longrightarrow \begin{array}{c} \text{free} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$ $\boxed{\vec{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$\underline{\lambda=2}$ $\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{pmatrix} \longrightarrow \begin{array}{c} \text{free} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$ $\boxed{\vec{V}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$

$\underline{\lambda=3}$ $\begin{pmatrix} -2 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{array}{c} \text{free} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$ $\boxed{\vec{V}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$

(Parts (c) and (d) on the reverse side)

(c) Find matrices P and D such that D is diagonal and

$$A = PDP^{-1}.$$

$D =$ diagonal w/ eigenvalues $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$P =$ change of basis from $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to std. basis

$= \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ (or any matrix where columns are multiples of these)

(d) Find an explicit formula for the matrix A^n.

(Your answer should be a 3×3 matrix, where each entry is written explicitly as a linear combination of nth powers of the eigenvalues of A).

we need P^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

so $P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

So $$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2^n & 2^n \\ 0 & 0 & 3^n \end{pmatrix}$$

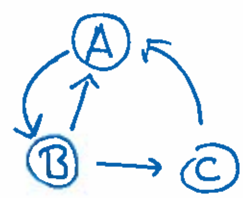
$$= \boxed{\begin{pmatrix} 1 & 1-2^n & 1-2^n \\ 0 & 2^n & 2^n-3^n \\ 0 & 0 & 3^n \end{pmatrix}}$$

8. [9 points] Your search engine is attempting to rank three webpages, denoted A, B, and C. You produce a sequence of scores for these pages, as follows. Each page is assigned an initial number of points in some manner. Then the scores are revised in a sequence of stages. In each stage, the following three things happen simultaneously:

- All of page A's points from the previous stage are given to page B.
- Page B's points from the previous stage are split evenly between pages A and C.
- All of page C's points from the previous stage are given to page A.

(a) Describe briefly how your search engine might have decided on these rules, based on the hyperlinks between pages A, B, and C (following the "original version" of Pagerank that we discussed in class).

A links only to B
B links to both A & C
C links only to A



(b) Suppose that in stage 1, the scores are as follows: A has 2 points, B has 4 points, and C has 1 point. What will the scores be in stage 3?

stage	1	2	3
A's score	2	3	3
B's score	4	2	3
C's score	1	2	1

A has 3
B has 3
C has 1

(Parts (c) and (d) on the reverse side)

(c) Write a transition matrix to describe how the scores are updated from one round to the next. More precisely: find a matrix T such that if $\vec{u}$ is a vector consisting of the scores of A, B and C at some stage, then $T\vec{u}$ is a vector consisting of the scores of A, B, and C in the following stage. (This was the "link matrix" in our discussion of Pagerank.)

$$T = \begin{pmatrix} 0 & 1/2 & 1 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

(d) Find an assignment of positive scores to A, B, and C such that, if these are taken to be the initial scores, then the scores of the three pages will not change from one stage to the next.

we want $T\vec{v} = \vec{v}$ (eigenvector of $\lambda = 1$).

$$\text{nullspace}(T-I) = \text{nullspace}\begin{pmatrix} -1 & 1/2 & 1 \\ 1 & -1 & 0 \\ 0 & 1/2 & -1 \end{pmatrix}$$

row-reducing $T-I$: $\longrightarrow \begin{pmatrix} 1 & -1/2 & -1 \\ 0 & -1/2 & 1 \\ 0 & 1/2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1/2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

free variable

$\Rightarrow \boxed{\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$ is one steady-state sol'n. (any positive multiple is ok).

(the steady-state probability vector is $\begin{pmatrix} 2/5 \\ 2/5 \\ 1/5 \end{pmatrix}$).

9. [9 points] Suppose that V is an inner product space, where the inner product of $\vec{v}$ and $\vec{w}$ is denoted $\langle \vec{v}, \vec{w} \rangle$. As usual, define $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$. Prove that for any two vectors $\vec{v}, \vec{w} \in V$,

$$\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 = 2\|\vec{v}\|^2 + 2\|\vec{w}\|^2.$$

$$\underline{\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2} = \langle \vec{v}+\vec{w}, \vec{v}+\vec{w} \rangle + \langle \vec{v}-\vec{w}, \vec{v}-\vec{w} \rangle$$

using linearity in both arguments

$$= \langle \vec{v}, \vec{v}+\vec{w} \rangle + \langle \vec{w}, \vec{v}+\vec{w} \rangle + \langle \vec{v}, \vec{v}-\vec{w} \rangle - \langle \vec{w}, \vec{v}-\vec{w} \rangle$$

$$= \langle \vec{v}, \vec{v} \rangle + \cancel{\langle \vec{v}, \vec{w} \rangle} + \cancel{\langle \vec{w}, \vec{v} \rangle} + \langle \vec{w}, \vec{w} \rangle$$

$$+ \langle \vec{v}, \vec{v} \rangle - \cancel{\langle \vec{v}, \vec{w} \rangle} - \cancel{\langle \vec{w}, \vec{v} \rangle} + \langle \vec{w}, \vec{w} \rangle$$

cancelling terms

$$= 2 \cdot \langle \vec{v}, \vec{v} \rangle + 2 \langle \vec{w}, \vec{w} \rangle$$

$$= \underline{2 \cdot \|\vec{v}\|^2 + 2 \cdot \|\vec{w}\|^2}, \quad \text{as desired.}$$

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10. [9 points] Suppose that $f(x)$ is a continuous function on the interval $[-\pi, \pi]$ that has been measured in a laboratory. You have reason to believe that the function should be approximately equal to some linear combination of $\sin(x)$ and $\cos(x)$. Using your experimental data, you compute the following integrals.

$$\int_{-\pi}^{\pi} f(x)\sin x\,dx = 7$$
$$\int_{-\pi}^{\pi} f(x)\cos x\,dx = 13$$

You may also use, without proof, the values of the following integrals.

$$\int_{-\pi}^{\pi} \sin^2 x\,dx = \pi$$
$$\int_{-\pi}^{\pi} \sin x\cos x\,dx = 0$$
$$\int_{-\pi}^{\pi} \cos^2 x\,dx = \pi$$

(a) Let $V = C^{(0)}[-\pi, \pi]$ denote the vector space of continuous functions on $[-\pi, \pi]$. Define an inner product on V, and identify each of the five integrals above as an inner product $\langle f, g\rangle$ of two elements f, g of V.

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

Then these five integrals are, respectively,

$$\langle f, \sin x\rangle = 7$$
$$\langle f, \cos x\rangle = 13$$
$$\langle \sin x, \sin x\rangle = \pi$$
$$\langle \sin x, \cos x\rangle = 0 \quad \leftarrow$$ ie. $\sin x$ & $\cos x$ are orthogonal.
$$\langle \cos x, \cos x\rangle = \pi$$

(b) Suppose that we choose two constants c_1, c_2 and define a function $g(x)$ as follows:

$$g(x) = c_1 \sin x + c_2 \cos x.$$

Find the values of c_1, c_2 that will ensure that $\int_{-\pi}^{\pi} (f(x) - g(x))^2 \, dx$ is as small as possible.

To minimize $\langle f-g, f-g \rangle$ $\left(= \int_{-\pi}^{\pi} (f(x)-g(x))^2 dx\right)$

it suffices to ensure that

$$\left.\begin{aligned} &\langle f-g, \sin x\rangle = 0 \\ \& \ &\langle f-g, \cos x\rangle = 0 \end{aligned}\right\}$$ proved on homework. (PSet 11, suppl. 2(b))

ie.

$$\langle f - c_1 \sin x - c_2 \cos x, \sin x\rangle = 0$$

$$\& \ \langle f - c_1 \sin x - c_2 \cos x, \cos x\rangle = 0.$$

These give: (by linearity)

$$0 = \langle f, \sin x\rangle - c_1 \langle \sin x, \sin x\rangle - c_2 \langle \cos x, \sin x\rangle$$

$$0 = 7 - c_1 \cdot \pi - c_2 \cdot 0$$

$$\Rightarrow \ \underline{c_1 = 7/\pi}$$

$$0 = \langle f, \cos x\rangle - c_1 \langle \sin x, \cos x\rangle - c_2 \langle \cos x, \cos x\rangle$$

$$0 = 13 - c_1 \cdot 0 - c_2 \cdot \pi$$

$$\Rightarrow \ \underline{c_2 = 13/\pi}$$

(can also use the "projection formula" from class).

so $$\boxed{g(x) = \frac{7}{\pi} \sin x + \frac{13}{\pi} \cos x}$$

is the best approximation (as measured by $\int_{-\pi}^{\pi} (f(x)-g(x))^2 dx$)