

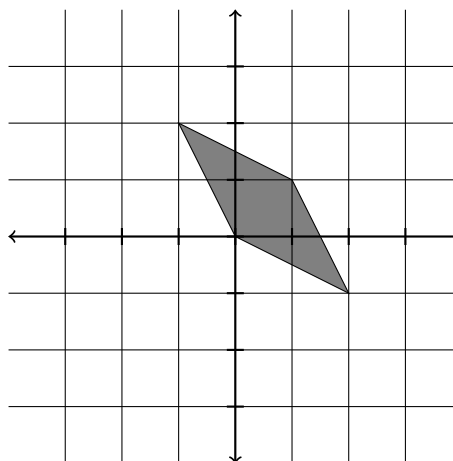
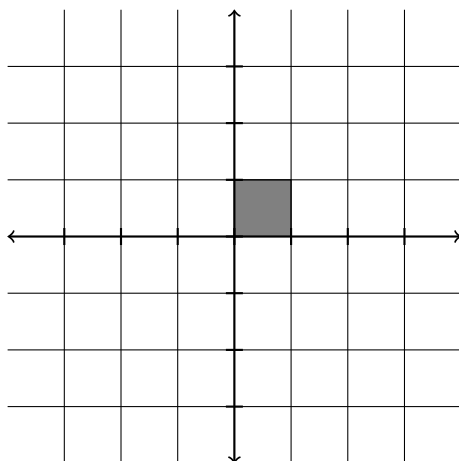
1. [12 points] For each choice of a matrix A and vector \vec{b} below, determine whether or not the linear system $A\vec{x} = \vec{b}$ is consistent. If it is consistent, find a solution. If it is inconsistent, find a “least-squares solution” (a vector \vec{x} such that $\|A\vec{x} - \vec{b}\|$ is as small as possible).

(a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$.

2. [9 points] Suppose that A is an $n \times n$ matrix and \vec{u}, \vec{v} are vectors in \mathbb{R}^n . Show that if $A\vec{u} = A\vec{v}$ and $\vec{u} \neq \vec{v}$, then A is not invertible.
3. [9 points] Suppose that A is a 2×2 matrix, and that A represents (in the standard basis) a graphics operation that transforms the unit square as shown. The figure on the left shows the original unit square, while the figure on the right shows its image after transformation.



- (a) Determine the matrix A .
- (b) Find a nonzero vector \vec{v} such that $A\vec{v} = \vec{v}$.

(Parts (c) and (d) on the reverse side)

- (c) Find a nonzero vector \vec{w} such that $A\vec{w}$ is a scalar multiple of \vec{w} , but is not equal to \vec{w} .
- (d) Let $B = \{\vec{v}, \vec{w}\}$, where \vec{v}, \vec{w} are the vectors found in parts (b) and (c). What is the matrix representation of this graphics operation in the basis B ?

4. [9 points] Let $\vec{u} = \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$. Find a basis of \mathbb{R}^3 that contains both \vec{u} and \vec{v} .

5. [9 points] Denote by \mathcal{P}_3 the vector space of polynomials in one variable of degree at most 3. Define a transformation $T : \mathcal{P}_3 \rightarrow \mathcal{P}_3$ by $T(p(x)) = \frac{d}{dx}p(x)$ for all $p(x) \in \mathcal{P}_3$. For example,

$$T(x^3 + 4x^2) = 3x^2 + 8x.$$

- (a) Show that T is a *linear* transformation.
- (b) Denote by $S = \{1, x, x^2, x^3\}$ the standard basis of \mathcal{P}_3 . Find $[T]_S$, the matrix representation of T in the standard basis.
- (c) What is the dimension of the null space of T ?
- (d) What is the dimension of the range of T ?
6. [9 points] Suppose that A is a square matrix, and n is a positive integer such that $A^n = \mathbf{0}$ (the all-0's matrix).
- (a) Prove that the only eigenvalue of A is $\lambda = 0$.
- (b) Prove that if A is nonzero, then A is *not* diagonalizable.

7. [12 points] Consider the matrix $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$.

- (a) What are the eigenvalues of A ?
- (b) Find an eigenvector for each eigenvalue that you found in part (a).
- (c) Find matrices P and D such that D is diagonal and

$$A = PDP^{-1}.$$

- (d) Find an explicit formula for the matrix A^n .

(Your answer should be a 3×3 matrix, where each entry is written explicitly as a linear combination of n th powers of the eigenvalues of A).

8. [9 points] Your search engine is attempting to rank three webpages, denoted A , B , and C . You produce a sequence of scores for these pages, as follows. Each page is assigned an initial number of points in some manner. Then the scores are revised in a sequence of stages. In each stage, the following three things happen simultaneously:
- All of page A 's points from the previous stage are given to page B .
 - Page B 's points from the previous stage are split evenly between pages A and C .
 - All of page C 's points from the previous stage are given to page A .

- (a) Describe briefly how your search engine might have decided on these rules, based on the hyperlinks between pages A , B , and C (following the “original version” of Pagerank that we discussed in class).
- (b) Suppose that in stage 1, the scores are as follows: A has 2 points, B has 4 points, and C has 1 point. What will the scores be in stage 3?
- (c) Write a transition matrix to describe how the scores are updated from one round to the next. More precisely: find a matrix T such that if \vec{u} is a vector consisting of the scores of A , B and C at some stage, then $T\vec{u}$ is a vector consisting of the scores of A , B , and C in the following stage. (This was the “link matrix” in our discussion of Pagerank.)
- (d) Find an assignment of positive scores to A , B , and C such that, if these are taken to be the initial scores, then the scores of the three pages will not change from one stage to the next.
9. [9 points] Suppose that V is an inner product space, where the inner product of \vec{v} and \vec{w} is denoted $\langle \vec{v}, \vec{w} \rangle$. As usual, define $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$. Prove that for any two vectors $\vec{v}, \vec{w} \in V$,

$$\|\vec{v} + \vec{w}\|^2 + \|\vec{v} - \vec{w}\|^2 = 2\|\vec{v}\|^2 + 2\|\vec{w}\|^2.$$

10. [9 points] Suppose that $f(x)$ is a continuous function on the interval $[-\pi, \pi]$ that has been measured in a laboratory. You have reason to believe that the function should be approximately equal to some linear combination of $\sin(x)$ and $\cos(x)$. Using your experimental data, you compute the following integrals.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin x \, dx &= 7 \\ \int_{-\pi}^{\pi} f(x) \cos x \, dx &= 13 \end{aligned}$$

You may also use, without proof, the values of the following integrals.

$$\begin{aligned} \int_{-\pi}^{\pi} \sin^2 x \, dx &= \pi \\ \int_{-\pi}^{\pi} \sin x \cos x \, dx &= 0 \\ \int_{-\pi}^{\pi} \cos^2 x \, dx &= \pi \end{aligned}$$

- (a) Let $V = \mathcal{C}^{(0)}[-\pi, \pi]$ denote the vector space of continuous functions on $[-\pi, \pi]$. Define an inner product on V , and identify each of the five integrals above as an inner product $\langle f, g \rangle$ of two elements f, g of V .
- (b) Suppose that we choose two constants c_1, c_2 and define a function $g(x)$ as follows:

$$g(x) = c_1 \sin x + c_2 \cos x.$$

Find the values of c_1, c_2 that will ensure that $\int_{-\pi}^{\pi} (f(x) - g(x))^2 \, dx$ is as small as possible.