1. [12 points] For each choice of a matrix A and vector $\vec{b}$ below, determine whether or not the linear system $A\vec{x} = \vec{b}$ is consistent. If it is consistent, find a solution. If it is inconsistent, find a "least-squares solution" (a vector $\vec{x}$ such that $\|A\vec{x} - \vec{b}\|$ is as small as possible).

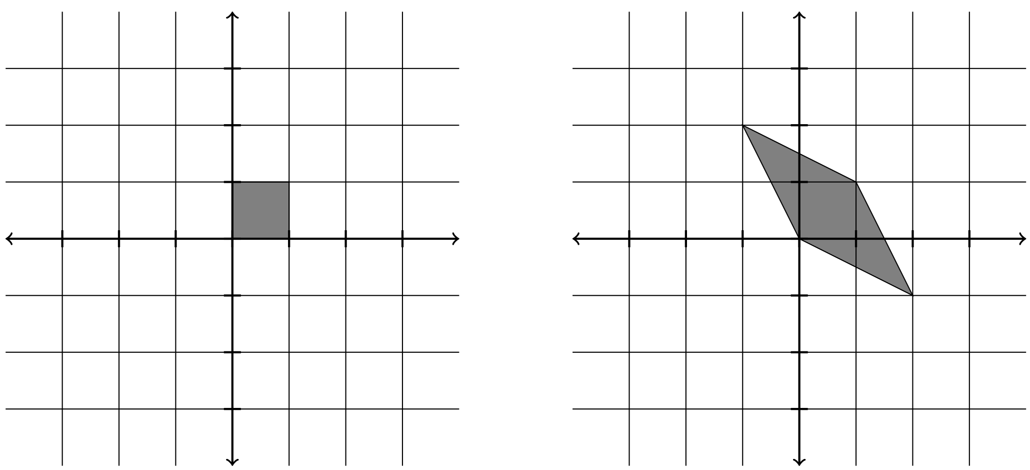
   (a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$.

   (b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$.

   (c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$.

2. [9 points] Suppose that A is an $n \times n$ matrix and $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$. Show that if $A\vec{u} = A\vec{v}$ and $\vec{u} \neq \vec{v}$, then A is not invertible.

3. [9 points] Suppose that A is a 2×2 matrix, and that A represents (in the standard basis) a graphics operation that transforms the unit square as shown. The figure on the left shows the original unit square, while the figure on the right shows its image after transformation.



   (a) Determine the matrix A.

   (b) Find a nonzero vector $\vec{v}$ such that $A\vec{v} = \vec{v}$.

   (c) Find a nonzero vector $\vec{w}$ such that $A\vec{w}$ is a scalar multiple of $\vec{w}$, but is not equal to $\vec{w}$.

   (d) Let $B = \{\vec{v}, \vec{w}\}$, where $\vec{v}, \vec{w}$ are the vectors found in parts (b) and (c). What is the matrix representation of this graphics operation in the basis B?

4. [9 points] Note (2025): this "expand to basis" technique is something we didn't emphasize this semester. Let $\vec{u} = \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$. Find a basis of $\mathbb{R}^3$ that contains both $\vec{u}$ and $\vec{v}$.

5. [9 points] Denote by $\mathcal{P}_3$ the vector space of polynomials in one variable of degree at most 3. Define a transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ by $T(p(x)) = \frac{d}{dx}p(x)$ for all $p(x) \in \mathcal{P}_3$. For example,

$$T(x^3 + 4x^2) = 3x^2 + 8x.$$

(a) Show that T is a *linear* transformation.

(b) Denote by $S = \{1, x, x^2, x^3\}$ the standard basis of $\mathcal{P}_3$. Find $[T]_S$, the matrix representation of T in the standard basis.

(c) What is the dimension of the null space of T?

(d) What is the dimension of the range of T?

6. [9 points] Suppose that A is a square matrix, and n is a positive integer such that $A^n = \mathbf{0}$ (the all-0's matrix).

(a) Prove that the only eigenvalue of A is $\lambda = 0$.

(b) Prove that if A is nonzero, then A is *not* diagonalizable.

7. [12 points] Consider the matrix $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) What are the eigenvalues of A?

(b) Find an eigenvector for each eigenvalue that you found in part (a).

(c) Find matrices P and D such that D is diagonal and

$$A = PDP^{-1}.$$

(d) Note (2025): we didn't emphasize this type of problem this semester, so I won't ask a question like this on our exam, but you do have the tools to figure it out. Find an explicit formula for the matrix A^n.
(Your answer should be a 3×3 matrix, where each entry is written explicitly as a linear combination of nth powers of the eigenvalues of A).

8. [9 points] Note (2025): we didn't cover this topic this semester. Your search engine is attempting to rank three webpages, denoted A, B, and C. You produce a sequence of scores for these pages, as follows. Each page is assigned an initial number of points in some manner. Then the scores are revised in a sequence of stages. In each stage, the following three things happen simultaneously:

- All of page A's points from the previous stage are given to page B.
- Page B's points from the previous stage are split evenly between pages A and C.
- All of page C's points from the previous stage are given to page A.

(a) Describe briefly how your search engine might have decided on these rules, based on the hyperlinks between pages A, B, and C (following the "original version" of Pagerank that we discussed in class).

(b) Suppose that in stage 1, the scores are as follows: A has 2 points, B has 4 points, and C has 1 point. What will the scores be in stage 3?

(c) Write a transition matrix to describe how the scores are updated from one round to the next. More precisely: find a matrix T such that if $\vec{u}$ is a vector consisting of the scores of A, B and C at some stage, then $T\vec{u}$ is a vector consisting of the scores of A, B, and C in the following stage. (This was the "link matrix" in our discussion of Pagerank.)

(d) Find an assignment of positive scores to A, B, and C such that, if these are taken to be the initial scores, then the scores of the three pages will not change from one stage to the next.

9. [9 points] Suppose that V is an inner product space, where the inner product of $\vec{v}$ and $\vec{w}$ is denoted $\langle \vec{v}, \vec{w} \rangle$. As usual, define $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$. Prove that for any two vectors $\vec{v}, \vec{w} \in V$,

$$\|\vec{v} + \vec{w}\|^2 + \|\vec{v} - \vec{w}\|^2 = 2\|\vec{v}\|^2 + 2\|\vec{w}\|^2.$$

10. [9 points] Suppose that $f(x)$ is a continuous function on the interval $[-\pi, \pi]$ that has been measured in a laboratory. You have reason to believe that the function should be approximately equal to some linear combination of $\sin(x)$ and $\cos(x)$. Using your experimental data, you compute the following integrals.

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin x \, dx &= 7 \\ \int_{-\pi}^{\pi} f(x) \cos x \, dx &= 13 \end{aligned}$$

You may also use, without proof, the values of the following integrals.

$$\begin{aligned} \int_{-\pi}^{\pi} \sin^2 x \, dx &= \pi \\ \int_{-\pi}^{\pi} \sin x \cos x \, dx &= 0 \\ \int_{-\pi}^{\pi} \cos^2 x \, dx &= \pi \end{aligned}$$

(a) Let $V = \mathcal{C}^{(0)}[-\pi, \pi]$ denote the vector space of continuous functions on $[-\pi, \pi]$. Define an inner product on V, and identify each of the five integrals above as an inner product $\langle f, g \rangle$ of two elements f, g of V.

(b) Suppose that we choose two constants c_1, c_2 and define a function $g(x)$ as follows:

$$g(x) = c_1 \sin x + c_2 \cos x.$$

Find the values of c_1, c_2 that will ensure that $\int_{-\pi}^{\pi} (f(x) - g(x))^2 \, dx$ is as small as possible.