Amherst College
Department of Mathematics and Statistics

MATH 272 FINAL EXAM SPRING 2018

NAME: Solutions

Read This First!

- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	9	9	9	12	12	9	9	9	12	90
Score:										

1. [9 points] When titanium tetrachlroide is sprayed into the air, it reacts with water vapor to form hydrogen chloride and fine particles of titanium dioxide (sometimes used to create smoke screens). The reaction can be expressed in the chemical equation

$$\underset{x_1}{}\,TiCl_4 + \underset{x_2}{H_2O} \longrightarrow \underset{x_3}{TiO_2} + \underset{x_4}{HCl}.$$

Write and solve a system of linear equations to balance this chemical equation.

one eq'n for each element:

Ti	$x_1 = x_3$
Cl	$4x_1 = x_4$
H	$2x_2 = x_4$
O	$x_2 = 2x_3$

ie

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 4 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Row-reducing gives:

$$\xrightarrow{R2 -= 4R1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & -2 & 0 \end{pmatrix} \xrightarrow[R2 \leftrightarrow R4]{R3 -= 2R4} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 4 & -1 \end{pmatrix}$$

$$\xrightarrow{\substack{R4 -= R3 \\ R2 += \frac{1}{2}R3 \\ R1 += \frac{1}{4}R3 \\ R3 *= 1/4}} \begin{pmatrix} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the gen'l sol'n is

$$\begin{pmatrix} 1/4 \cdot x_4 \\ 1/2 \cdot x_4 \\ 1/4 \cdot x_4 \\ x_4 \end{pmatrix} \quad (x_4 \text{ free}).$$

The smallest integer sol'n is $(x_4 = 4)$

$$1 \cdot TiCl_4 + 2 \cdot H_2O \longrightarrow 1 \cdot TiO_2 + 4 \cdot HCl.$$

2. [9 points] Suppose that there are two cities, A and B. Every year, 20% of the citizens of city A relocate to city B, and 10% of the citizens of city B relocate to city A.

(a) Viewing the movement of population between these cities as a Markov process, find the *transition matrix* T of this process. More explicitly, T should be a matrix with the following property: if a and b are the current populations of cities A and B, then the two coordinates of $T\begin{pmatrix} a \\ b \end{pmatrix}$ are the populations of A and B next year.

$$T = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$$

(b) What percentage of the current population of A will live in city A two years later?

$$T^2 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}\begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$$

$$= \begin{pmatrix} 0.64 + 0.02 & 0.08 + 0.09 \\ 0.16 + 0.18 & 0.02 + 0.81 \end{pmatrix}$$

$$= \begin{pmatrix} 0.66 & 0.17 \\ 0.34 & 0.83 \end{pmatrix}$$

So $\boxed{66\%}$ of this year's pop. in A will be in A after two years.

(64% stay the whole time, while 2% move to B then move back).

(continued on reverse)

(c) Find the steady-state probability vector for this process.

$T\vec{s} = \vec{s}$

$\vec{s} \in N(T-I)$

$$T-I = \begin{pmatrix} -0.2 & 0.1 \\ 0.2 & -0.1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$$

so gen'l sol'n to $(T-I)\vec{x} = \vec{0}$ is $\begin{pmatrix} \frac{1}{2}y \\ y \end{pmatrix}$ (y free).

The prob. vector in this set is the one where $\frac{1}{2}y + y = 1$ ie. $y = \frac{2}{3}$.

$$\boxed{\vec{s} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}}.$$

3. [9 points] Suppose that V and W are two vector spaces of the *same dimension*, and that $T: V \to W$ is a linear transformation.

(a) Prove that if T is one-to-one, then T is also onto.

T one-to-one $\Rightarrow \dim N(T) = 0$

$\Rightarrow \dim R(T) = \dim V$ (since $\dim V = \dim R(T) + \dim N(T)$ by Rank-nullity)

$\Rightarrow \dim R(T) = \dim W$ (since $\dim V = \dim W$)

$\Rightarrow T$ is onto.

(b) Prove, conversely, that if T is onto, then T is one-to-one.

T onto $\Rightarrow \dim R(T) = \dim W$

$\Rightarrow \dim R(T) = \dim V$ (since $\dim V = \dim W$)

$\Rightarrow \dim N(T) + \dim V = \dim V$ (since $\dim N(T) + \dim R(T) = \dim V$ & $\dim R(T) = \dim V$)

$\Rightarrow \dim N(T) = 0$

$\Rightarrow T$ is one-to-one.

(continued on reverse)

(c) Prove that if V and W have *different* dimensions, then it is impossible for T to be both one-to-one and onto.

We prove the contrapositive: if T is both one-to-one & onto (ie., an isomorphism), then $\dim V = \dim W$.

Assume T is both one-to-one & onto.

Then $\dim N(T) = 0$

and $\dim R(T) = \dim W$.

So since $\dim N(T) + \dim R(T) = \dim V$ (Rank-nullity),

it follows that $0 + \dim W = \dim V$

ie. $\dim W = \dim V$.

Hence if $\dim W \neq \dim V$, then T cannot be both one-to-one & onto.

4. [12 points] Denote by $\mathcal{P}_2$ the vector space of polynomials of degree at most 2.

(a) Consider the basis $B = \{x+1, x^2+x, x^2+1\}$ of $\mathcal{P}_2$. Find the coordinates $[x^2]_B$ of x^2 in the basis B.

$$c_1(x+1) + c_2(x^2+x) + c_3(x^2+1) = x^2$$

$$\Leftrightarrow \begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right) \quad \text{so} \quad [x^2]_B = \boxed{\begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}}$$

(b) Let $S = \{1, x, x^2\}$ denote the standard basis of $\mathcal{P}_2$, and let B be the basis from part (a). Determine the two change of basis matrices $[I]_B^S$ and $[I]_S^B$.

$[I]_B^S$ is easier to see by inspection:

$$[x+1]_S = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad [x^2+x]_S = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad [x^2+1]_S = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{so} \quad \boxed{[I]_B^S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.}$$

Now, $[I]_S^B$ is the inverse.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right)$$

$$\text{so} \quad \boxed{[I]_S^B = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{pmatrix}}$$

(continued on reverse)

(c) Consider the linear operator $T: \mathcal{P}_2 \to \mathcal{P}_2$ given by

$$T(p(x)) = x \cdot p'(x)$$

(where $p'(x)$ denotes the derivative of $p(x)$). Find the matrix representation $[T]_S$ of T in the standard basis S.

$$[T(1)]_S = [x \cdot 0]_S = [0]_S = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[T(x)]_S = [x \cdot 1]_S = [x]_S = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[T(x^2)]_S = [x \cdot 2x] = [2x^2]_S = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

so $\boxed{[T]_S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$

(d) What is the matrix representation $[T]_B$ of T in the basis B? **You may express your answer in terms of your answers to parts (b) and (c)**, without simplifying any matrix multiplication.

$$[T]_B = [I]_S^B [T]_S [I]_B^S$$

$$= \underline{\begin{pmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}$$

(this evaluates to

$$\begin{pmatrix} 1/2 & -1/2 & -1 \\ 1/2 & 3/2 & 1 \\ -1/2 & 1/2 & 1 \end{pmatrix}.)$$

5. [12 points] Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$.

(a) Determine the eigenvalues of A.

char. eqn: $\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)(4-\lambda) + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-2)(\lambda-3) = 0$

$\boxed{\lambda_1 = 2,\ \lambda_2 = 3}$

(b) For each eigenvalue, find a nonzero eigenvector.

$V_2 = N\begin{pmatrix} 1-2 & 1 \\ -2 & 4-2 \end{pmatrix} = N\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} = N\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$= \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$; let $\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.}$

$V_3 = N\begin{pmatrix} 1-3 & 1 \\ -2 & 4-3 \end{pmatrix} = N\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} = N\begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}$

$= \text{span}\left\{\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right\}$. let $\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

(continued on reverse)

(c) Diagonalize the matrix A (that is, determine matrices P and D such that $A = PDP^{-1}$, where D is a diagonal matrix).

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$ (change of basis from $\{\vec{v}_1, \vec{v}_2\}$ to standard)

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$ (eigenvalues on diagonal; this is the matrix rep. in basis $\{\vec{v}_1, \vec{v}_2\}$).

(d) Find an explicit formula for A^n. In your answer, each of the four entries of A^n should be given as an explicit formula of n.

$$\begin{aligned} A^n &= P \cdot D^n \cdot P^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^n & 3^n \\ 2^n & 2 \cdot 3^n \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 2 \cdot 2^n - 3^n & -2^n + 3^n \\ 2 \cdot 2^n - 2 \cdot 3^n & -2^n + 2 \cdot 3^n \end{pmatrix}} \end{aligned}$$

or, alternatively, $\boxed{2^n \cdot \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} + 3^n \cdot \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}}$

6. [9 points] Consider the following matrix.

$$A = \begin{pmatrix} 1 & 1 & -2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 1 \\ 1 & 2 & -7 & 7 & 10 \\ 1 & 3 & -12 & 0 & 4 \end{pmatrix}$$

You may use, without proof, that the matrix row-reduces to the following matrix (in reduced echelon form).

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find a basis for the span of the columns of A.

The RREF has pivots in columns 1, 2, & 4, so these columns give a basis for the span in the original matrix.

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \\ 0 \end{pmatrix} \right\}}.$$

(b) Find a basis for the null space of A.

Gen'l sol'n to $A\vec{x} = 0$ is

$$\begin{aligned} x_1 &= -3x_3 - x_5 \\ x_2 &= 5x_3 - x_5 \\ x_3 &\text{ free} \\ x_4 &= -x_5 \\ x_5 &\text{ free} \end{aligned}$$

in vector form:

$$\begin{pmatrix} -3x_3 - x_5 \\ 5x_3 - x_5 \\ x_3 \\ -x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

so $\boxed{\left\{ \begin{pmatrix} -3 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$ is a basis for $N(A)$.

(continued on reverse)

(c) Find the general solution (in terms of one or more free variables) of the matrix equation

$$A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Hint: The right side of this equation is equal to the first column of A. Use this to find one specific solution, and then deduce the general solution from this.

one specific sol'n is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, since $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is the 1st column of A.

So the gen'l sol'n is given by this specific sol'n ~~that~~ plus a solution to $A\vec{x}=\vec{0}$, ie

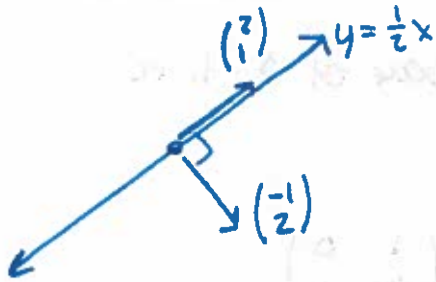
$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}$$

w/ x_3, x_5 free

or, in coordinates,

$$\boxed{\begin{aligned} x_1 &= 1 - 3x_3 - x_5 \\ x_2 &= 5x_3 - x_5 \\ x_3 &\ \text{free} \\ x_4 &= -x_5 \\ x_5 &\ \text{free} \end{aligned}}$$

7. [9 points] (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by reflection across the line $y = \frac{1}{2}x$. Find the matrix representation of T in the standard basis.



we can choose another basis where T is easier to write down:

$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (on the axis)

$\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (perpendicular to the axis)

so $T\vec{v}_1 = \vec{v}_1$ & $T\vec{v}_2 = -\vec{v}_2$.

So in basis $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$, we have

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence

$$[T]_S = [I]^S_{\mathcal{B}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} [I]^{\mathcal{B}}_S$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \boxed{\frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}}$$

(continued on reverse)

(b) Let $A = \frac{1}{13}\begin{pmatrix} 5 & +12 \\ 12 & -5 \end{pmatrix}$. This is the matrix representation of reflection across some line through the origin in $\mathbb{R}^2$ (the "axis of reflection."). What is the axis of reflection?

The axis must be the eigenspace of $\lambda = 1$. ie the nullspace of

$$\frac{1}{13}\begin{pmatrix} 5 & +12 \\ 12 & -5 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -8/13 & +12/13 \\ 12/13 & -18/13 \end{pmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -3/2 \\ 0 & 0 \end{pmatrix}$$

so V_1 is spanned by $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$. ie. it is the line $\boxed{y = \frac{2}{3}x}$

8. [9 points] Consider the following three points in the plane:

$$(x_1, y_1) = (1, 2), \qquad (x_2, y_2) = (3, 2), \qquad (x_3, y_3) = (3, 1).$$

Suppose that we wish to find a line of best fit for these three points. This problem concerns two ways to do this, which optimize different choices of "error function."

(a) Suppose we wish to find the coefficients of a line in the form $y = c_1 x + c_2$ that minimizes

$$\sum_{i=1}^{3} (c_1 x_i + c_2 - y_i)^2. \quad = \left\| c_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\|^2$$

Determine three vectors $\vec{v}_1, \vec{v}_2, \vec{b}$ such that this problem is equivalent to minimizing

$$\|c_1 \vec{v}_1 + c_2 \vec{v}_2 - \vec{b}\|.$$

Let

$$\vec{v}_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix},$$

(b) Find a linear system of equations whose solution gives the optimal choice of c_1, c_2 in the previous part. **You do not need to solve the system. It is sufficient to write the equations.**

The normal eq'n for this least squares problem is

$$\begin{pmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \cdot \vec{b} \\ \vec{v}_2 \cdot \vec{b} \end{pmatrix}$$

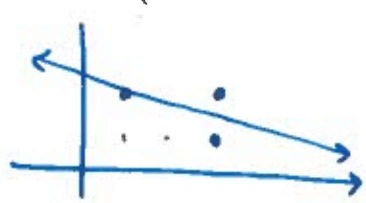
ie.

$$\boxed{\begin{pmatrix} 19 & 7 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \end{pmatrix}}$$

not necessary to write but: the solution is

$c_1 = -1/4$, $c_2 = 9/4$

& the line is $y = -\frac{1}{4}x + \frac{9}{4}$



(continued on reverse)

(c) Now suppose that we wish to find the coefficients of a line in the form $x = c_1 y + c_2$ that minimize

$$\sum_{i=1}^{3} (c_1 y_i + c_2 - x_i)^2.$$

Find a linear system of equations whose solution gives the optimal choice of c_1 and c_2 in this case. **You do not need to solve the system.**

Like before, we find $\vec{v}_1, \vec{v}_2, \vec{b}$:

$$\vec{v}_1 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

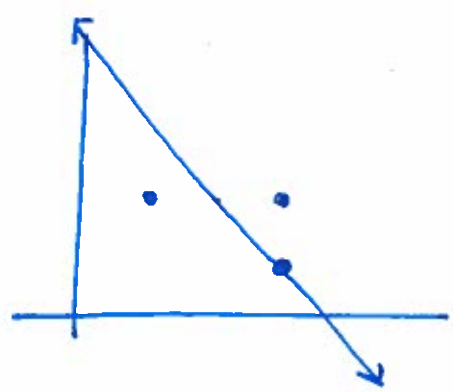
& the normal eq'n is

$$\begin{pmatrix} \vec{v}_1 \cdot \vec{v}_1 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & \vec{v}_2 \cdot \vec{v}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \cdot \vec{b} \\ \vec{v}_2 \cdot \vec{b} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 9 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}}$$

If solved, this gives $c_1 = -1$, $c_2 = 4$

so the line is $x = -y + 4$, ie. $y = -x + 4$.



9. [12 points] Suppose that $B = \{\vec{v}_1, \cdots \vec{v}_n\}$ is a basis for a vector space V, and $T: V \to W$ is a linear transformation.

(a) Prove that T is one-to-one if and only if $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent.

Suppose that T is one-to-one.

Then for all $c_1, \cdots, c_n$ such that $\sum_{i=1}^{n} c_i T(\vec{v}_i) = \vec{0}$,

it follows that $T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \vec{0}$ (T is linear),

hence $\sum_{i=1}^{n} c_i \vec{v}_i = \vec{0}$, since T is one-to-one.

Since B is a basis, it is LI, so it follows that $c_1 = c_2 = \cdots = c_n = 0$.

Hence $T(\vec{v}_1), \cdots, T(\vec{v}_n)$ are LI.

Conversely suppose that $T(\vec{v}_1), \cdots, T(\vec{v}_n)$ are LI.

For any $\vec{v} \in N(T)$, we can write $\vec{v} = \sum_{i=1}^{n} c_i \vec{v}_i$ (since B spans V), hence $T(\vec{v}) = \vec{0} \Rightarrow \sum_{i=1}^{n} c_i T(\vec{v}_i) = 0$ (T is linear).

Since ~~T is on~~ $T(\vec{v}_1), \cdots, T(\vec{v}_n)$ are LI, it follows that all c_i are 0, so $\vec{v} = \vec{0}$ as well. So $N(T)$ contains only $\vec{0}$, ie. T is one-to-one.

(b) **True of False** (no explanation needed): the statement in part (a) remains true if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is only assumed to be linearly independent (not necessarily a basis).

$\boxed{\text{False}}$ (observe: the proof above used both that B is LI & that it spans V.

A counterexample if it doesn't span V: let $V = \mathbb{R}^2$, $B = \{\begin{pmatrix}1\\0\end{pmatrix}\}$, & $T(\vec{x}) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\vec{x}$)

(continued on reverse)

(c) Prove that T is onto if and only if $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ spans W.

Suppose T is onto.

Then for all $\vec{w} \in W$, there is some $\vec{v} \in V$ s.t. $T(\vec{v}) = \vec{w}$ (T is onto). Since $\mathcal{B}$ spans V, there exist $c_1, \cdots, c_n$ with $\vec{v} = \sum c_i \vec{v}_i$.

Then $\vec{w} = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i T(\vec{v}_i)$ (T is linear)

so $\vec{w} \in \text{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$.

Hence all $\vec{w} \in W$ are in this span, so $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ spans W.

Conversely, suppose $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ spans W.

Then for all $\vec{w} \in W$, $\exists\ c_1, \cdots, c_n$ st. $\vec{w} = \sum_{i=1}^{n} c_i T(\vec{v}_i)$.

So $\vec{w} = T\left(\sum_{i=1}^{n} c_i \vec{v}_i\right)$, so $\vec{w} \in R(T)$.

Hence all $\vec{w} \in W$ are in $R(T)$, ie. T is onto.

(d) **True of False** (no explanation needed): the statement in part (c) remains true if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is only assumed to span V (but is not necessarily a basis).

$\boxed{\text{True.}}$ The proof above used that $\mathcal{B}$ spans V, but not that $\mathcal{B}$ is LI.