1. [9 points] When titanium tetrachlroide is sprayed into the air, it reacts with water vapor to form hydrogen chloride and fine particles of titanium dioxide (sometimes used to create smoke screens). The reaction can be expressed in the chemical equation

$$\mathrm{Ti\,Cl_4} + \mathrm{H_2O} \longrightarrow \mathrm{TiO_2} + \mathrm{HCl}.$$

Write and solve a system of linear equations to balance this chemical equation.

2. [9 points] Suppose that there are two cities, A and B. Every year, 20% of the citizens of city A relocate to city B, and 10% of the citizens of city B relocate to city A.

   (a) Viewing the movement of population between these cities as a Markov process, find the *transition matrix* T of this process. More explicitly, T should be a matrix with the following property: if a and b are the current populations of cities A and B, then the two coordinates of $T\begin{pmatrix} a \\ b \end{pmatrix}$ are the populations of A and B next year.

   (b) What percentage of the current population of A will live in city A two years later?

   (c) Note (2025): we didn't use this vocabulary this semester. Find the steady-state probability vector for this process.

3. [9 points] Suppose that V and W are two vector spaces of the *same dimension*, and that $T: V \to W$ is a linear transformation.

   (a) Prove that if T is one-to-one, then T is also onto.

   (b) Prove, conversely, that if T is onto, then T is one-to-one.

   (c) Prove that if V and W have *different* dimensions, then it is impossible for T to be both one-to-one and onto.

4. [12 points] Denote by $\mathcal{P}_2$ the vector space of polynomials of degree at most 2.

   (a) Consider the basis $B = \{x+1, x^2+x, x^2+1\}$ of $\mathcal{P}_2$. Find the coordinates $[x^2]_B$ of x^2 in the basis B.

   (b) Let $S = \{1, x, x^2\}$ denote the standard basis of $\mathcal{P}_2$, and let B be the basis from part (a). Determine the two change of basis matrices $[I]_B^S$ and $[I]_S^B$.

   (c) Consider the linear operator $T: \mathcal{P}_2 \to \mathcal{P}_2$ given by

   $$T(p(x)) = x \cdot p'(x)$$

   (where $p'(x)$ denotes the derivative of $p(x)$). Find the matrix representation $[T]_S$ of T in the standard basis S.

   (d) What is the matrix representation $[T]_B$ of T in the basis B? **You may express your answer in terms of your answers to parts (b) and (c)**, without simplifying any matrix multiplication.

5. [12 points] Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$.

   (a) Determine the eigenvalues of A.

   (b) For each eigenvalue, find a nonzero eigenvector.

(c) Diagonalize the matrix A (that is, determine matrices P and D such that $A = PDP^{-1}$, where D is a diagonal matrix).

(d) Find an explicit formula for A^n. In your answer, each of the four entries of A^n should be given as an explicit formula of n.

6. [9 points] Consider the following matrix.

$$A = \begin{pmatrix} 1 & 1 & -2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 1 \\ 1 & 2 & -7 & 7 & 10 \\ 1 & 3 & -12 & 0 & 4 \end{pmatrix}$$

You may use, without proof, that the matrix row-reduces to the following matrix (in reduced echelon form).

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Find a basis for the span of the columns of A.

(b) Find a basis for the null space of A.

(c) Find the general solution (in terms of one or more free variables) of the matrix equation

$$A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Hint: The right side of this equation is equal to the first column of A. Use this to find one specific solution, and then deduce the general solution from this.

7. [9 points] (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by reflection across the line $y = \frac{1}{2}x$. Find the matrix representation of T in the standard basis.

(b) Let $A = \frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$. This is the matrix representation of reflection across some line through the origin in $\mathbb{R}^2$ (the "axis of reflection."). What is the axis of reflection?

8. [9 points] Consider the following three points in the plane:

$$(x_1, y_1) = (1, 2), \qquad (x_2, y_2) = (3, 2), \qquad (x_3, y_3) = (3, 1).$$

Suppose that we wish to find a line of best fit for these three points. This problem concerns two ways to do this, which optimize different choices of "error function."

(a) Suppose we wish to find the coefficients of a line in the form $y = c_1 x + c_2$ that minimizes

$$\sum_{i=1}^{3} (c_1 x_i + c_2 - y_i)^2.$$

Determine three vectors $\vec{v}_1, \vec{v}_2, \vec{b}$ such that this problem is equivalent to minimizing

$$\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|.$$

(b) Find a linear system of equations whose solution gives the optimal choice of c_1, c_2 in the previous part. **You do not need to solve the system. It is sufficient to write the equations.**

(c) Now suppose that we wish to find the coefficients of a line in the form $x = c_1 y + c_2$ that minimize

$$\sum_{i=1}^{3} (c_1 y_i + c_2 - x_i)^2.$$

Find a linear system of equations whose solution gives the optimal choice of c_1 and c_2 in this case. **You do not need to solve the system.**

9. [12 points] Suppose that $B = \{\vec{v}_1, \cdots \vec{v}_n\}$ is a basis for a vector space V, and $T: V \to W$ is a linear transformation.

(a) Prove that T is one-to-one if and only if $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent.

(b) **True of False** (no explanation needed): the statement in part (a) remains true if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is only assumed to be linearly independent (not necessarily a basis).

(c) Prove that T is onto if and only if $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ spans W.

(d) **True of False** (no explanation needed): the statement in part (c) remains true if $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is only assumed to span V (but is not necessarily a basis).