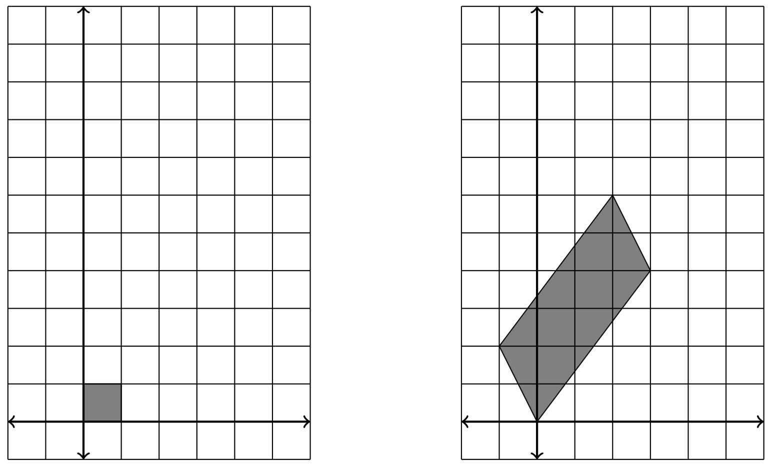
1. Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that has the following effect on the unit square.



   (a) [2 points] Find the matrix representation of T in the standard basis.
   (There are two possible answers based on how you interpret the picture. You need only give one.)

   (b) [2 points] Determine $R(T)$ and $N(T)$ (no explanation is necessary for this part).

   (c) [3 points] Find the matrix representation, in the standard basis, of the inverse transformation T^{-1}.

   (d) [3 points] Determine a point $\vec{x} \in \mathbb{R}^2$ that is sent to $\begin{pmatrix} 5 \\ 10 \end{pmatrix}$ by this transformation (i.e. $T(\vec{x}) = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$).

2. Let A be the following 3×5 matrix.

$$A = \begin{pmatrix} 2 & -6 & -9 & -11 & -8 \\ 2 & -6 & -6 & -8 & -4 \\ 1 & -3 & -3 & -4 & 0 \end{pmatrix}$$

The reduced row echelon form of A is as follows (you do not need to verify this yourself).

$$\begin{pmatrix} 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   (a) [3 points] Give a basis for $N(A)$.

   (b) [3 points] Give a basis for $R(A)$.

   (c) [3 points] Let A be the same matrix above, and let $\vec{b}$ be any vector in $\mathbb{R}^3$. Explain why the matrix equation $A\vec{x} = \vec{b}$ is consistent, regardless of the choice of $\vec{b}$. How many free variables occur in the general solution of $A\vec{x} = \vec{b}$?

3. [9 points] Let

$$A = \begin{pmatrix} 2 & -2 & -3 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}.$$

Determine the inverse matrix A^{-1}.

4. Suppose that $T : V \to W$ is a linear transformation of vector spaces, and $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a set of three vectors in V.

   (a) [5 points] Suppose $\{T(\vec{v}_1),\ T(\vec{v}_2),\ T(\vec{v}_3)\}$ is a linearly independent set in W. Prove that S is a linearly independent set in V.

   (b) [5 points] Prove that if S is a linearly independent set in V and T is one-to-one, then $\{T(\vec{v}_1),\ T(\vec{v}_2),\ T(\vec{v}_3)\}$ is a linearly independent set in W.

5. [9 points] Denote by $\vec{u}, \vec{v}, \vec{b}$ the following three vectors in $\mathbb{R}^4$.

$$\vec{u} = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0 \\ -11 \\ -11 \\ 11 \end{pmatrix}$$

   Determine the linear combination $\vec{w}$ of $\{\vec{u}, \vec{v}\}$ that is closest to $\vec{b}$ (that is, the linear combination that minimizes $\|\vec{w} - \vec{b}\|$).

6. Let V be an inner product space. Suppose that W is a subspace of V with basis $B = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

   (a) [2 points] What is the dimension of W?

   (b) [4 points] Suppose that $\vec{b}$ is a vector in V, and $\vec{u}$ is a vector in W such that

$$\vec{w}_1 \perp (\vec{b} - \vec{u}), \qquad \vec{w}_2 \perp (\vec{b} - \vec{u}), \qquad \text{and} \qquad \vec{w}_3 \perp (\vec{b} - \vec{u}).$$

   Prove that $\vec{b} - \vec{u}$ is orthogonal to every vector in W.

   (c) [4 points] Let $\vec{b}, \vec{u}$ be as in part (b). Prove that if $\vec{v}$ is any other vector in W, then

$$\|\vec{b} - \vec{v}\|^2 = \|\vec{b} - \vec{u}\|^2 + \|\vec{u} - \vec{v}\|^2.$$

7. Define a map $T : \mathcal{P}_2 \to \mathbb{R}^4$ by the formula

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{pmatrix}.$$

   T is a linear transformation (you do not need to prove this).

   (a) [4 points] Let $B = \{(x-1)(x-2),\ x(x-2),\ x(x-1)\}$. This is a basis of $\mathcal{P}_2$ (you do not need to prove this). Let S denote the standard basis of $\mathbb{R}^4$. Determine the matrix representation $[T]_B^S$ of T with respect to the basis B of $\mathcal{P}_2$ and the standard basis S of $\mathbb{R}^4$.

   (b) [3 points] Let A be the matrix $[T]_B^S$ obtained in the previous part. Prove that $\vec{b} \in R(T)$ if and only if the matrix equation $A\vec{x} = \vec{b}$ is consistent.

(c) [3 points] Suppose that we are given four constants a, b, c, d, and wish to find a polynomial $p(x) \in \mathcal{P}_2$ whose graph passes through the four points $(0, a), (1, b), (2, c), (3, d)$. For which values of a, b, c, d is this possible? Express your answer as a linear equation in a, b, c, and d.

Hint: interpret this as asking when a linear system is consistent, using the previous part.

8. [9 points] Let A be an $n \times n$ matrix, and λ a scalar. Define as usual the following set (called the *eigenspace* in class).

$$V_\lambda = \{\vec{v} \in \mathbb{R}^n : \; A\vec{v} = \lambda\vec{v}\}$$

Prove that V_λ is a *subspace* of $\mathbb{R}^n$.

9. In a certain town, 8000 customers subscribe to internet service from company A. A competitor, company B, enters the market and begins to draw customers from company A.

Suppose that every year, 10% of company A's customers change their service to company B, and 30% of company B's customers change their service to company A.

For example:

- In the first year, 800 customers (10% of 8000) change service from A and B, after which company A has 7200 customers and company B has 800 customers.
- In the second year, 720 customers (10% of 7200) switch from A to B, and 240 customers (30% of 800) switch from B to A. So after two years, company A has $7200 - 720 + 240 = 6720$ customers and company B has $800 + 720 - 240 = 1280$ customers.

(a) [2 points] Find a 2×2 matrix M encoding the change in number of customers of each company from one year to the next. More precisely: if a, b denote the number of customers of companies A and B (respectively) in a given year, and a', b' denote the number of customers in the following year, the matrix M should satisfy

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = M \begin{pmatrix} a \\ b \end{pmatrix}.$$

You can check your answer by verifying that

$$\begin{pmatrix} 7200 \\ 800 \end{pmatrix} = M \begin{pmatrix} 8000 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6720 \\ 1280 \end{pmatrix} = M^2 \begin{pmatrix} 8000 \\ 0 \end{pmatrix}.$$

(b) [6 points] Find the eigenvalues of M, and an eigenvector for each eigenvalue.

(c) [3 points] Express $\begin{pmatrix} 8000 \\ 0 \end{pmatrix}$ as a linear combination of the eigenvectors you found in part (b).

(d) [3 points] Find a formula for $M^n \begin{pmatrix} 8000 \\ 0 \end{pmatrix}$, and use your formula to evaluate $\lim\limits_{n\to\infty} M^n \begin{pmatrix} 8000 \\ 0 \end{pmatrix}$.