1. [9 points] Evaluate the **determinant** of the following matrix.

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{pmatrix}$$

2. [9 points] Define a linear operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ by the following formula.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 9x - 4y \\ 25x - 11y \end{bmatrix}$$

   (a) Let S denote the standard basis of $\mathbb{R}^2$. Determine the matrix representation $[T]_S$.

   (b) Let $B = \left\{ \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$. This is a basis for $\mathbb{R}^2$ (you do not need to prove this). Determine the matrix representation $[T]_B$.

3. [9 points] Define three vectors in $\mathbb{R}^4$ as follows.

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 5 \\ 5 \\ 6 \\ 0 \end{pmatrix}$$

   Find the linear combination of $\vec{u}$ and $\vec{v}$ that is as close as possible to $\vec{b}$.

4. [9 points] Define a linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^4$ by $T(\vec{v}) = A\vec{v}$, where A is the following matrix.

$$A = \begin{pmatrix} 1 & -2 & -1 & 2 \\ 1 & -2 & 0 & 5 \\ 1 & -2 & 0 & 5 \\ 1 & -2 & 1 & 8 \end{pmatrix}$$

   (a) Find a basis for the **range** $R(T)$. What is its dimension?

   (b) Find a basis for the **null space** $N(T)$. What is its dimension?

5. [9 points] Let A be the following matrix.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & -1 & 3 \end{pmatrix}$$

   (a) Determine the eigenvalues of A.

   (b) Find a corresponding eigenvector for each eigenvalue.

   (c) Diagonalize the matrix A. That is: determine a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

6. [9 points] Let V be an inner product space, and let $\vec{u}, \vec{b} \in V$ be two specific vectors in V.

(a) Recall the following definition: the projection of $\vec{b}$ onto $\vec{u}$ is

$$\text{proj}_{\vec{u}}\vec{b} = \frac{\langle \vec{u}, \vec{b} \rangle}{\langle \vec{u}, \vec{u} \rangle}\vec{u}.$$

Prove, using this definition, that $(\vec{b} - \text{proj}_{\vec{u}}\vec{b}) \perp \vec{u}$.

(b) Use part (a) to prove that for any constant c,

$$\|\vec{b} - c\vec{u}\|^2 = \|\vec{b} - \text{proj}_{\vec{u}}\vec{b}\|^2 + \|c\vec{u} - \text{proj}_{\vec{u}}\vec{b}\|^2$$

(c) Use part (b) to prove that the projection $\text{proj}_{\vec{u}}\vec{b}$ is closer to $\vec{b}$ than any other multiple of $\vec{u}$.

7. [9 points] Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across the line $y = 8x$.

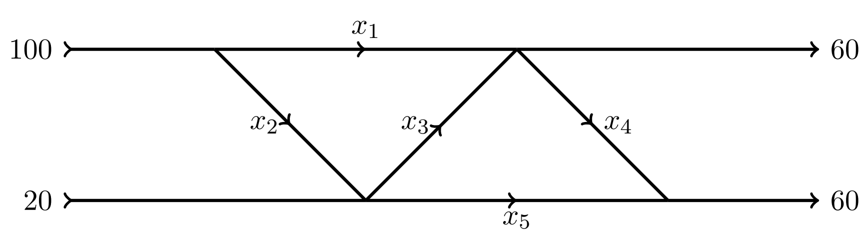
(a) Find a nonzero vector $\vec{u}$ such that $T(\vec{u}) = \vec{u}$ (i.e. an eigenvector for eigenvalue $\lambda = 1$). You can do this without any computation; think geometrically about T.

(b) Find a nonzero vector $\vec{v}$ such that $T(\vec{v}) = -\vec{v}$ (in other words, an eigenvector for eigenvalue $\lambda = -1$). Again, you can do this without much computation; it's useful to think about the dot product with $\vec{u}$.

(c) Let $B = \{\vec{u}, \vec{v}\}$, where these are the vectors you found in parts (a), (b). This is a basis for $\mathbb{R}^2$ (you don't need to prove this). Find $[T]_B$ (this should not require any computations at all; just use the equations $T(\vec{u}) = \vec{u}$ and $T(\vec{v}) = -\vec{v}$).

(d) Compute $[I]_B^S$, and use it to compute $[T]_S$ (here, S is the standard basis for $\mathbb{R}^2$).

8. [9 points] Consider the traffic flow pattern shown below. The diagram is interpreted as follows: 100 cars enter along the inbound road in the upper left, and 20 cars enter along the road in the lower left. Along both of the outbound roads on the right, 60 cars exit. The traffic along the five road segments in the middle are denoted by variables x_1, x_2, x_3, x_4, x_5.



(a) Write a system of linear equations in x_1, x_2, x_3, x_4, x_5 describing the traffic flow in this network.

(b) Find the general solution to this system of linear equations.
Hint: your answer should involve two free variables.

(c) You found in part (b) that there are infinitely many possible traffic patterns in this network. In reality, some patterns are more plausible than others. For example, drivers will tend to drive on less busy roads, causing the traffic to balance itself across the various roads.

One way to model this is to assume that the drivers will choose the traffic pattern than minimizes the quantity

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2.$$

Among all possible solutions found in part (b), determine which solution minimizes this quantity. Your answer should be a specific choice of values x_1, x_2, x_3, x_4, x_5.

Hint: The quantity to be minimized is the same as $\|\vec{x}\|^2$, and you have expressed $\vec{x}$ in terms of two free variables. You can convert this into a least-squares problem.

9. [9 points] Let $T : V \to W$ be a linear transformation.

   (a) Prove that if T is injective, then $N(T) = \{\vec{0}\}$.

   (b) Prove the converse: if $N(T) = \{\vec{0}\}$, then T is injective.

   (c) Prove that if T is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$, then T cannot be injective.