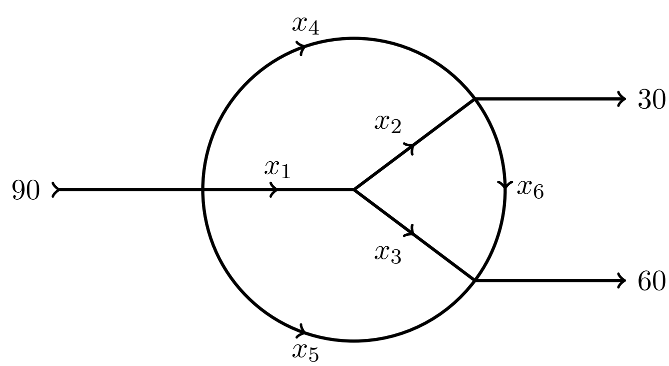
1. [12 points] Find the determinant of the following 4×4 matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix}$$

2. [12 points] Consider the traffic flow pattern shown below. The diagram is interpreted as follows: 90 cars enter along the inbound road on the left. Along the outbound roads on the right, 30 cars exit along the top road, and 60 cars exist along the bottom road. The traffic along the six road segments in the middle are denoted by variables $x_1, x_2, x_3, x_4, x_5, x_6$ as shown.



   (a) Write a system of linear equations in $x_1, x_2, x_3, x_4, x_5, x_6$ describing the traffic flow in this network.

   (b) Find the general solution to this system of linear equations.
   Hint: your answer should involve three free variables.

   (c) You found in part (b) that there are infinitely many possible traffic patterns in this network. In reality, some patterns are more plausible than others. For example, drivers will tend to drive on less busy roads, causing the traffic to balance itself across the various roads.

   One way to model this is to assume that the drivers will choose the traffic pattern than minimizes the quantity
   $$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2.$$

   Among all possible solutions found in part (b), determine which solution minimizes this quantity. Your answer should be a specific choice of values $x_1, x_2, x_3, x_4, x_5, x_6$.
   Hint: The quantity to be minimized is the same as $\|\vec{x}\|^2$, and you have expressed $\vec{x}$ in terms of three free variables. You can convert this into a least-squares problem.

3. [12 points] Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 4 & 5 \\ 2 & -5 & 9 & -13 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

   (a) Find a *basis* for the nullspace $N(T)$.

   (b) What is the *dimension* of the range $R(T)$?

4. [12 points] Let $T : V \to V$ be a linear operator. Suppose that $\vec{u}, \vec{v}, \vec{w}$ are three vectors in V such that $\{T(\vec{u}), T(\vec{v}), T(\vec{w})\}$ is a basis of V. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is also a basis of V.

5. [12 points] Consider the 3×3 matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$.

   (a) Find the eigenvalues of A.

   (b) For each eigenvalue, you found in part (a), find a basis for the corresponding eigenspace.

   (c) Diagonalize A. That is, state an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. **You do not need to do any new computation in this problem; it is possible to write this down using the results from part (a) and part (b).**

   (d) Find an explicit formula for the vector $A^n \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$, where n is a positive integer. Your answer should be stated in terms of nth powers of the eigenvalues you found in part (a).

6. [12 points] Define a linear transformation $T : \mathcal{P}_2 \to \mathbb{R}^3$ by

$$T(p(x)) = \begin{pmatrix} p(1) \\ p'(1) \\ p''(1) \end{pmatrix}.$$

   (a) Let $B = \{1, x, x^2\}$ be the standard basis of $\mathcal{P}_2$, and let $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis of $\mathbb{R}^3$. Find the matrix representation $[T]_B^S$ of T with respect to the bases B and S.

   (b) Let $B' = \{1, x-1, (x-1)^2\}$ be a different basis of $\mathcal{P}_2$. Find the matrix representation $[T]_{B'}^S$ of T with respect to the bases B' and S.

   (c) Determine the rank and nullity of T.

7. [12 points] A function $f(t)$ on the interval $[-1, 1]$ is measured in a lab, and the graph of the signal looks like it is approximated well by a quadratic function. In order to do so, you compute the following three integrals:

$$\int_{-1}^{1} f(t)\, dt = 36$$
$$\int_{-1}^{1} f(t)\, t\, dt = 2$$
$$\int_{-1}^{1} f(t)\, t^2\, dt = 20$$

   Determine the quadratic function $p(t) = a + bt + ct^2$ that best approximates $f(t)$ in the least-squares sense. That is, determine the coefficients a, b, c such that $\int_{-1}^{1} [f(t) - p(t)]^2\, dt$ is minimized.

8. [12 points] Suppose that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the following two equations.

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = 2\begin{bmatrix}1\\1\end{bmatrix}$$
$$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = -\begin{bmatrix}-1\\1\end{bmatrix}$$

(a) Let $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$. This is a basis of $\mathbb{R}^2$ (you do not need to prove this). Determine the matrix representation $[T]_B$ of T with respect to the basis B.

(b) Let S be the standard basis of $\mathbb{R}^2$. Determine the change of basis matrices $[I]_B^S$ and $[I]_S^B$.

(c) Determine the matrix representation $[T]_S$ of T with respect to the standard basis S.

9. [12 points] Suppose that $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4, \vec{u}_5$ are given vectors in $\mathbb{R}^3$. Assume that:

$$\vec{u}_3 \in \text{span}\{\vec{u}_1, \vec{u}_2\}.$$

Prove that

$$\vec{u}_5 \in \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$$

if and only if

$$\vec{u}_5 \in \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_4\}.$$