1. [9 points] Solve the following system of linear equations.

$$\begin{array}{rcrcrcrcr} & & x_2 & + & x_3 & + & x_4 & = & 5 \\ x_1 & & & + & 3x_3 & + & 7x_4 & = & 14 \\ x_1 & & & + & 2x_3 & + & 5x_4 & = & 11 \end{array}$$

Row-reducing the aug. matrix:

$$\begin{array}{r} \\ \\ -1\times \end{array}\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 5 \\ 1 & 0 & 3 & 7 & 14 \\ 1 & 0 & 2 & 5 & 11 \end{array}\right]$$

$$\downarrow$$

$$\begin{array}{r} \\ \\ +3\times \quad +1\times \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 3 & 7 & 14 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & -1 & -2 & -3 \end{array}\right]\begin{array}{l} \\ \\ ]\times(-1) \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$\underbrace{\quad\quad\quad}_{x_1, x_2, x_3 \text{ bound}}$ $\underbrace{\quad}_{x_4 \text{ free}}$

$$\boxed{\begin{aligned} x_1 &= 5 - t \\ x_2 &= 2 + t \\ x_3 &= 3 - 2t \\ x_4 &= t \end{aligned}}$$

2. [9 points]
Recall that two matrices A, B *commute* if $AB = BA$. Consider the following three matrices.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

(a) Determine whether A and B commute.

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\neq$$
$$BA = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$\boxed{\text{no}}$

(b) Determine whether B and C commute.

$$BC = \begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$$
$$=$$
$$CB = \begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$$

$\boxed{\text{yes}}$

(c) Determine whether A and C commute.

$$AC = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$$
$$\neq$$
$$CA = \begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}$$

$\boxed{\text{no}}$

3. [9 points]

(a) Suppose that A is an $n \times n$ matrix. Show that if A is invertible, then $A\vec{x} = \vec{0}$ has no nontrivial solutions $\vec{x}$.

Given A^{-1} exists:

If $A\vec{x} = \vec{0}$

then $A^{-1}A\vec{x} = A^{-1}\vec{0}$

$\Rightarrow \vec{x} = \vec{0}$. (since $A^{-1}A = I$ & $A^{-1}\vec{0} = \vec{0}$).

So any sol'n is the trivial one. ie. there are no nontrivial sol'ns.

(b) Using part (a), show that $A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & -3 & 1 \\ -1 & -1 & 2 \end{pmatrix}$ is not an invertible matrix. (*Hint:* the numbers in each row sum to 0.)

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \vec{A}_1 + 1 \cdot \vec{A}_2 + 1 \cdot \vec{A}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $A\vec{x} = \vec{0}$ has a nontrivial sol'n, namely $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

By (a), this is impossible if A is invertible. So A must not be invertible.

4. [9 points] The augmented matrix of a linear system has the form

$$\begin{array}{cc} -2\times & \\ +1\times & \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 2 & 3 & -2 & b \\ -1 & -1 & 1 & c \end{array}\right).$$

(a) Determine the values of a, b, c for which this linear system in consistent.

row-reducing gives

$$\begin{array}{c} +2\times \\ +1\times \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & 0 & b-2a \\ 0 & 1 & 0 & a+c \end{array}\right] \begin{array}{c} \\ ]\times(-1) \\ \\ \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -3a+2b \\ 0 & 1 & 0 & 2a-b \\ 0 & 0 & 0 & -a+b+c \end{array}\right]$$

Which is consistent iff the last row is all 0's. i.e. $\boxed{a=b+c}$.

(b) For those values of a, b, c for which the system is consistent, does it have a unique solution or infinitely many solutions? Briefly explain why.

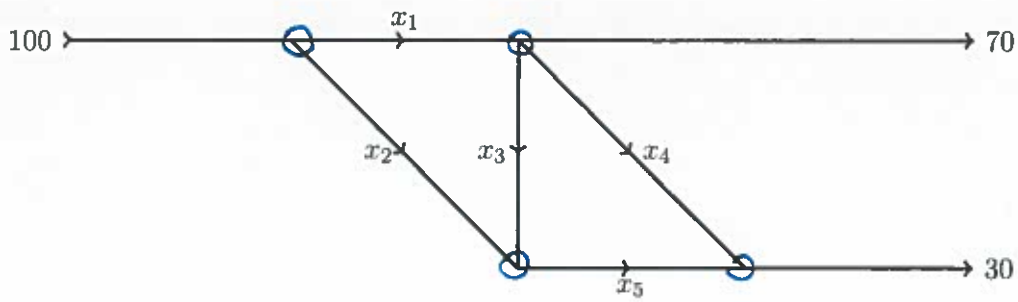
There are ∞ sol'ns (when consistent) because x_3 will be a free variable (no pivot in column 3).

(In fact, the sol'n is

$$x_1 = -3a+2b+t$$
$$x_2 = 2a-b$$
$$x_3 = t$$

in this case).

5. [9 points] Write a system of linear equations that describes the traffic flow pattern for the network in the figure. **You do not need to solve the system.**



traffic in = traffic out

$$\begin{cases} 100 = x_1 + x_2 \\ x_1 = x_3 + x_4 + 70 \\ x_2 + x_3 = x_5 \\ x_4 + x_5 = 30 \end{cases}$$

or, written in the usual way,

$$\begin{cases} x_1 + x_2 = 100 \\ x_1 - x_3 - x_4 = 70 \\ x_2 + x_3 - x_5 = 0 \\ x_4 + x_5 = 30 \end{cases}$$

⑥ Consider the following three vectors.

$$\vec{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

(a) Show that the only ~~set of vectors~~ choice of constants c_1, c_2, c_3 such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

is $c_1 = c_2 = c_3 = 0$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \iff \begin{pmatrix}1&1&1\\1&2&1\\1&3&2\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$ Row-reducing:

$$\left(\begin{array}{ccc|c}1&1&1&0\\1&2&1&0\\1&3&2&0\end{array}\right) \xrightarrow[R3 -= R1]{R2 -= R1} \left(\begin{array}{ccc|c}1&1&1&0\\0&1&0&0\\0&2&1&0\end{array}\right) \xrightarrow[R3 -= 2R2]{R1 -= \cancel{2}\cdot R2} \left(\begin{array}{ccc|c}1&0&1&0\\0&1&0&0\\0&0&1&0\end{array}\right)$$

$$\xrightarrow{R1 -= R3} \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right).$$ This shows that the only sol'n is $c_1 = c_2 = c_3 = 0$, as desired.

(b) Find scalars such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{pmatrix}2\\1\\3\end{pmatrix}$.

We row-reduce using the same steps, now w/ $\begin{pmatrix}2\\1\\3\end{pmatrix}$ on the right.

$$\left(\begin{array}{ccc|c}1&1&1&2\\1&2&1&1\\1&3&2&3\end{array}\right) \xrightarrow[R3 -= R1]{R2 -= R1} \left(\begin{array}{ccc|c}1&1&1&2\\0&1&0&-1\\0&2&1&1\end{array}\right)$$

$$\xrightarrow[R3 -= 2\cdot R2]{R1 -= R2} \left(\begin{array}{ccc|c}1&0&1&3\\0&1&0&-1\\0&0&1&3\end{array}\right) \xrightarrow{R1 -= R3} \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&-1\\0&0&1&3\end{array}\right)$$

So the (unique) solution is $\boxed{c_1 = 0, \ c_2 = -1, \ c_3 = 3.}$