1. [9 points] Solve the following linear system of equations.

$$\begin{array}{rrrrcl} & 4x_2 & +12x_3 & -7x_4 & = & 8 \\ & x_2 & +3x_3 & -2x_4 & = & 1 \\ x_1 & -2x_2 & -4x_3 & +5x_4 & = & 9 \end{array}$$

$$\left(\begin{array}{cccc|c} 0 & 4 & 12 & -7 & 8 \\ 0 & \cancel{2}1 & 3 & -2 & 1 \\ 1 & -2 & -4 & 5 & 9 \end{array}\right) \xrightarrow[R3 -= 4R2]{R1 \leftrightarrow R3,\ R1 += 2R2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 11 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\xrightarrow[R2 += 2R3]{R1 -= R3} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 7 \\ 0 & 1 & 3 & 0 & 9 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\boxed{\begin{array}{l} x_1 = 7 - 2x_3 \\ x_2 = 9 - 3x_3 \\ x_3 \text{ free} \\ x_4 = 4 \end{array}}$$

2. [9 points] Describe all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that commute with the matrix $A = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$.

Note: There are infinitely many such matrices; one way to express your answer is to express a, b, c, d in terms of one or more free variables.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} 3a-4b & 4a+3b \\ 3c-4d & 4c+3d \end{pmatrix} = \begin{pmatrix} 3a+4c & 3b+4d \\ -4a+3c & -4b+3d \end{pmatrix}$$

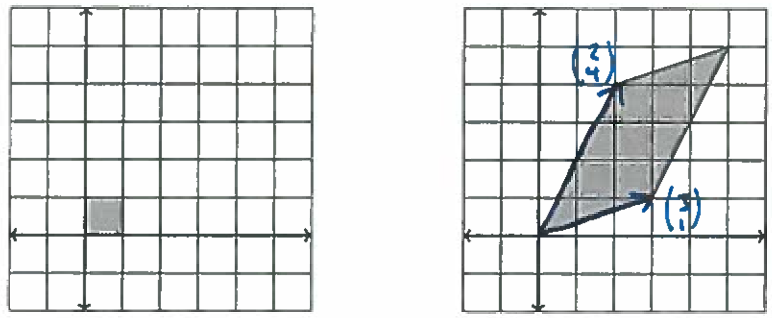
$$\Longleftrightarrow \quad \left\{ \begin{aligned} 3a-4b &= 3a+4c \\ 4a+3b &= 3b+4d \\ 3c-4d &= -4a+3c \\ 4c+3d &= -4b+3d \end{aligned} \right. \qquad \text{all hold}$$

$$\Longleftrightarrow \quad \left\{ \begin{aligned} -4b - 4c &= 0 \\ 4a - 4d &= 0 \\ +4a - 4d &= 0 \\ 4b+4c &= 0 \end{aligned} \right. \qquad \text{all hold}$$

$$\Longleftrightarrow \quad \underline{b = -c \quad \text{and} \quad a = d.}$$

So the matrices that commute with A are those of the form $\begin{pmatrix} d & -c \\ c & d \end{pmatrix}$, where c, d are free.

3. [9 points] Suppose that A is a 2×2 matrix that transforms the unit square in the plane in the manner shown below.



(a) Determine A. There are more than one possible answer; you only need to give one.

A sends $\binom{1}{0}$ & $\binom{0}{1}$ to $\binom{3}{1}$ & $\binom{2}{4}$ (in some order), so the columns of A are $\binom{3}{1}$ & $\binom{2}{4}$ in some order.

Either $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ or $\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ is a valid answer.

(b) Using the matrix A you found in part (a), determine the area of the parallelogram in the second picture.

The parallelogram has area $|\det A|$,

which is either $|3\cdot 4 - 2\cdot 1| = 10$

or $|2\cdot 1 - 3\cdot 4| = 10$

(same answer regardless of the answer chosen in (a), of course).

4. [9 points]

(a) Find the determinants of the following two matrices.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 5 \\ 1 & 0 & 9 \\ 2 & -1 & 14 \end{pmatrix}$$

$$\det A = \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \quad (R2 -= R1 \ \& \ R3 -= R1)$$

$$= 1 \cdot 1 \cdot 2 \quad \text{(upper triangular)}$$

$$= \boxed{2.}$$

(or use the 3x3 formula: $1\cdot3\cdot5 + 2\cdot7\cdot1 + 3\cdot1\cdot2 - 1\cdot7\cdot2 - 2\cdot1\cdot5 - 3\cdot3\cdot1$
$= 15 + 14 + 6 - 14 - 10 - 9 = 2$).

$$\det B = \det\begin{pmatrix} 1 & -1 & 5 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{pmatrix} \quad (R2 -= R1 \ \& \ R3 -= 2R1)$$

$$= \det\begin{pmatrix} 1 & -1 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad (R3 -= R2)$$

$$= \boxed{0} \quad \text{(there's a row of 0's).}$$

(Part (b) on reverse side)

(b) One of the matrices A, B is invertible, while the other one is not. Determine which is which, and then compute the inverse of the one that is invertible. For convenience, the two matrices are printed again below.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 1 & 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 5 \\ 1 & 0 & 9 \\ 2 & -1 & 14 \end{pmatrix}$$

A is the invertible one, since $\det A \neq 0$.

Row-reducing alongside the identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 7 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} R2 -= R1 \\ R3 -= R1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} R1 -= 2R2 \\ R3 *= \frac{1}{2} \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 3 & -2 & 0 \\ 0 & 1 & 4 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

$$\begin{array}{l} R2 -= 4R3 \\ R1 += 5R3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -2 & 5/2 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right)$$

so

$$\boxed{A^{-1} = \begin{pmatrix} 1/2 & -2 & 5/2 \\ 1 & 1 & -2 \\ -1/2 & 0 & 1/2 \end{pmatrix}}$$

5. [9 points] Consider the following four vectors.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 8 \\ -3 \\ -1 \end{pmatrix}$$

Express $\vec{b}$ as a linear combination of $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ -1 \end{pmatrix}$$

solving by row-reduction:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & -1 \end{array}\right) \xrightarrow[R3 -= R1]{R2 -= R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & -2 & 1 & -11 \\ 0 & 0 & 3 & -9 \end{array}\right)$$

$$\xrightarrow{R3 *= \frac{1}{3}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & -2 & 1 & -11 \\ 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow[R2 -= R3]{R1 -= R3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 11 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\xrightarrow[R1 -= R2]{R2 *= (-\frac{1}{2})} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

so $c_1 = 7$, $c_2 = 4$, $c_3 = -3$ is the unique sol'n.

$$\boxed{\vec{b} = 7\vec{v}_1 + 4\vec{v}_2 - 3\vec{v}_3}$$

(6) Suppose that A & B are invertible $n \times n$ matrices, and that the product AB is invertible. Prove that A is also invertible.

<u>solution</u>

Let C be the inverse of AB. This means that

$$(AB)C = I.$$

By associativity, $A(BC) = I$ as well. This means that BC is the inverse of A; in particular, A is invertible.