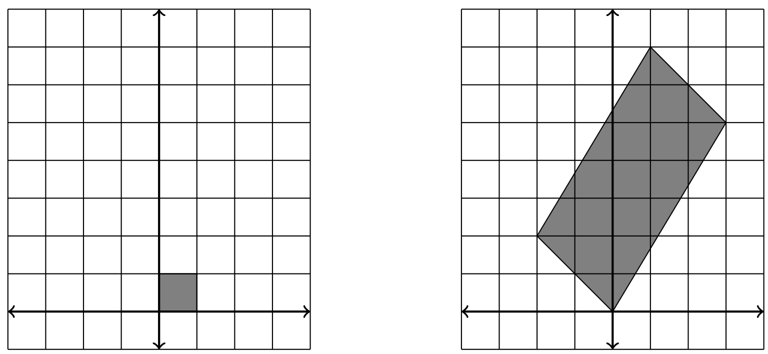
1. [9 points] Solve the following linear system of equations (your answer should describe all possible solutions, using one or more free variables if necessary).

$$\begin{array}{rrrrcr} & x_2 & -2x_3 & & = & 5 \\ x_1 & +x_2 & +3x_3 & +x_4 & = & 15 \\ -x_1 & +x_2 & -7x_3 & & = & 2 \end{array}$$

2. [9 points] Suppose that A is a 2×2 matrix that transforms the unit square in the plane in the manner shown below.



   (a) Determine A. There are more than one possible answer; you only need to give one.

   (b) Using the matrix A you found in part (a), determine the area of the parallelogram in the second picture.

3. [9 points] Determine all values of λ for which the following 4×4 matrix is *not* invertible.

$$\begin{pmatrix} 0 & \lambda & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & 5 & 7 & 5 \end{pmatrix}$$

4. [9 points] Describe the set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that can be expressed as linear combinations of the two vectors $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$. Your answer should take the form of an equation in terms of x, y, and z.

5. [9 points] (This problem asks you to prove two statements that were proved in class. It is not sufficient to state that these were proved in class; you should provide complete proofs.) Suppose that A is an $m \times n$ matrix (not necessarily square), $\vec{b}$ is a vector in $\mathbb{R}^m$, and $\vec{u}$ is a vector in $\mathbb{R}^n$ such that $A\vec{u} = \vec{b}$.

   (a) Prove that if $\vec{u}$ is the *unique* solution to the matrix equation $A\vec{x} = \vec{b}$, then the homogeneous equation $A\vec{x} = \vec{0}$ has no *nontrivial* solutions.

   (b) Prove, conversely, that if the homogeneous equation $A\vec{x} = \vec{0}$ has no nontrivial solutions, then $\vec{u}$ is the *unique* solution to the (inhomogeneous) matrix equation $A\vec{x} = \vec{b}$.