Amherst College
Department of Mathematics and Statistics

MATH 272 | MIDTERM 1 | SPRING 2019

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back.
- No calculators or other devices are permitted.
- You may use any of the blank pages to continue answers if you run out of space. Please clearly indicate on the problem's original page if you do so, so that I know to look for it.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	Total
Points:	9	9	9	9	9	45
Score:						

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1. [9 points] Solve the following linear system of equations (your answer should describe all possible solutions).

$$\begin{array}{rrrrrcl} & & 3x_3 & +9x_4 & +x_5 & = & 1 \\ -x_1 & +8x_2 & +x_3 & +8x_4 & & = & 0 \\ x_1 & -8x_2 & +x_3 & -2x_4 & +x_5 & = & 0 \end{array}$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 3 & 9 & 1 & 1 \\ -1_{+1} & 8_{-8} & 1_{+1} & 8_{-2} & 0_{+1} & 0_{+0} \\ 1 & -8 & 1 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R1 \leftrightarrow R3]{R2 += R3} \left(\begin{array}{ccccc|c} 1 & -8 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2/2 & 6/2 & 1/2 & 0 \\ 0 & 0 & 3 & 9 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R2 *= \frac{1}{2}} \left(\begin{array}{ccccc|c} 1 & -8 & 1_{-1} & -2_{-3} & 1_{-1/2} & 0 \\ 0 & 0 & 1 & 3 & 1/2 & 0 \\ 0 & 0 & 3_{-3} & 9_{-9} & 1_{-3/2} & 1 \end{array}\right)$$

$$\xrightarrow[R3 -= 3R2]{R1 -= R2} \left(\begin{array}{ccccc|c} 1 & -8 & 0 & -5 & 1/2_{+1/2} & 0_{+1} \\ 0 & 0 & 1 & 3 & 1/2_{+1/2} & 0_{+1} \\ 0 & 0 & 0 & 0 & -1/2 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R2 += R3 \\ R3 *= (-2)}]{R1 += R3} \left(\begin{array}{ccccc|c} 1 & -8 & 0 & -5 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

gen'l sol'n:

$$\boxed{\begin{array}{l} x_1 = 1 + 8x_2 + 5x_4 \\ x_2 \text{ free} \\ x_3 = 1 - 3x_4 \\ x_4 \text{ free} \\ x_5 = -2 \end{array}}$$

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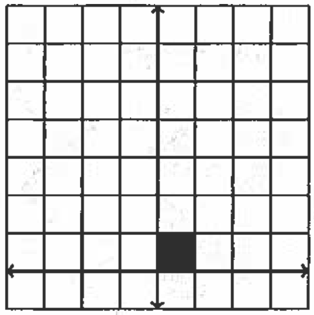
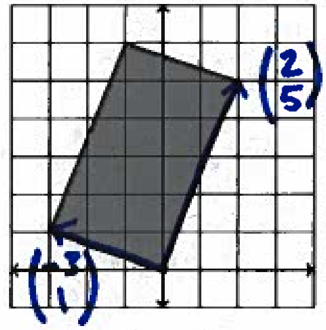
2. [9 points] Suppose that A is a 2×2 matrix that transforms the unit square in the plane in the manner shown below.





(a) Determine A. There are more than one possible answer; you only need to give one.

$\binom{1}{0}$ & $\binom{0}{1}$ are sent to $\binom{2}{5}$ & $\binom{-3}{1}$ (in one order or the other).

A could be either $\boxed{\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix}}$ or $\boxed{\begin{pmatrix} -3 & 2 \\ 1 & 5 \end{pmatrix}}$.

(b) Using the matrix A you found in part (a), determine the area of the parallelogram in the second picture.

$$\det\begin{pmatrix} 2 & -3 \\ 5 & 1 \end{pmatrix} = 2\cdot 1 - (-3)\cdot 5$$
$$= 2 + 15$$
$$= \boxed{17}$$

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3. [9 points] When titanium tetrachlroide is sprayed into the air, it reacts with water vapor to form hydrogen chloride and fine particles of titanium dioxide (sometimes used to create smoke screens). The reaction can be expressed in the chemical equation

$$\underset{c_1}{TiCl_4} + \underset{c_2}{H_2O} \longrightarrow \underset{c_3}{TiO_2} + \underset{c_4}{HCl}.$$

Write ~~and solve a~~ system of linear equations to balance this chemical equation.
Note: because this is a relatively simple reaction, it is certainly possible to balance this equation rapidly by other techniques, but for purposes of this problem you should carefully show the linear system involved.

Ti	c_1	=	c_3
Cl	$4c_1$	=	c_4
H	$2c_2$	=	c_4
O	c_2	=	$2c_3$

as an aug. matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{array}\right)$$

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4. [9 points] Define $\vec{u}$ and $\vec{v}$ to be the following two vectors. $\vec{u} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix}$

(a) Express $\begin{pmatrix} 2 \\ -11 \\ -5 \end{pmatrix}$ as a linear combination of $\vec{u}$ and $\vec{v}$.

$$\left(\begin{array}{cc|c} -1 & 1 & 2 \\ 1 & 8 & -11 \\ 1 & 2 & -5 \end{array}\right) \xrightarrow[R3 += R1]{R2 += R1} \left(\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 9 & -9 \\ 0 & 3 & -3 \end{array}\right)$$

$$\xrightarrow{R2 *= 1/9} \left(\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{array}\right) \xrightarrow[R3 -= 3R2]{R1 -= R2} \left(\begin{array}{cc|c} -1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R1 *= (-1)} \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$c_1 = -3$ & $c_2 = -1$.

$$\boxed{\begin{pmatrix} 2 \\ -11 \\ -5 \end{pmatrix} = -3 \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 8 \\ 2 \end{pmatrix}}$$

(b) Show that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is not equal to a linear combination of $\vec{u}$ and $\vec{v}$.

$$\left(\begin{array}{cc|c} -1 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 9 & 1 \\ 0 & 3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 1 & 1/9 \\ 0 & 3 & \underset{-3/9}{1} \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 1 & 1/9 \\ 0 & 0 & 2/3 \end{array}\right)$$

last eq'n is $0 = 2/3$

$\Longrightarrow$ this system is inconsistent

$\Longrightarrow$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is not an LC of $\vec{u}$ & $\vec{v}$.

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5. [9 points] Suppose that $\vec{u}, \vec{v}, \vec{w}, \vec{b}$ are four vectors in $\mathbb{R}^n$, such that

- $\vec{b}$ is a linear combination of $\vec{u}, \vec{v}, \vec{w}$, and
- $\vec{v}$ is a linear combination of $\vec{u}$ and $\vec{w}$.

(a) Prove that $\vec{b}$ is equal to a linear combination of $\vec{u}$ and $\vec{w}$ alone (not involving $\vec{v}$).

Since $\vec{b}$ is a LC of $\vec{u}, \vec{v}, \vec{w}$, $\exists\, c_1, c_2, c_3$ st.

$$\vec{b} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}.$$

Since $\vec{v}$ is a LC of $\vec{u}$ & $\vec{w}$, $\exists\, d_1, d_2$ st.

$$\vec{v} = d_1\vec{u} + d_2\vec{w}.$$

Thus $\vec{b} = c_1\vec{u} + c_2(d_1\vec{u} + d_2\vec{w}) + c_3\vec{w}$

$= (c_1 + c_2 d_1)\vec{u} + (c_2 d_2 + c_3)\vec{w}$

$\Rightarrow$ $\vec{b}$ is a LC of $\vec{u}$ & $\vec{w}$.

(b) Prove that there are infinitely many ways to write $\vec{b}$ as a linear combination of $\vec{u}, \vec{v}$, and $\vec{w}$.

For any scalar t,

$$\vec{b} = c_1\vec{u} + c_2\vec{v} + c_3\vec{w}$$

& $\vec{0} = d_1 t\vec{u} - t\cdot\vec{v} + d_2 t\vec{w}$

$\Rightarrow$ (adding the equations)

$$\vec{b} = (c_1 + d_1 t)\vec{u} + (c_2 - t)\vec{v} + (c_3 + d_2 t)\vec{w}$$

since there are infinitely many choices for t, this gives infinitely many ways to write $\vec{b}$ as a LC of $\vec{u}, \vec{v}$, & $\vec{w}$.

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