Amherst College
Department of Mathematics and Statistics

MATH 272 | MIDTERM 2 | FALL 2017

NAME: Solutions

Read This First!

- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. You may use the backs of pages for additional work space.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	Total
Points:	9	9	9	9	9	9	54
Score:							

1. **Short answer questions.** No explanations are necessary.

(a) [3 points] Give a basis for each of the following vector spaces.

- $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $M_{2\times 2}$ (the space of 2×2 matrices)

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

- $\mathcal{P}_2$ (the space of polynomials of degree at most 2)

$$\{1, x, x^2\}$$

(b) [3 points] Suppose that B and B' are two bases for a vector space V, and the change of basis matrix is

$$[I]_B^{B'} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

Let $\vec{v} \in V$ be a vector such that $[\vec{v}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Determine $[\vec{v}]_{B'}$.

$$\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 7 \end{bmatrix}}$$

(c) [3 points] Suppose that $T : \mathbb{R}^5 \to \mathbb{R}^3$ is a linear transformation, and that T is surjective (onto). Determine $\dim N(T)$ (where $N(T)$ denotes the null space of T).

$$R(T) = \mathbb{R}^3$$

$$\Rightarrow \dim N(T) = \dim \mathbb{R}^5 - \dim \mathbb{R}^3$$

$$= \boxed{2}$$

2. [9 points] Consider the following three vectors.

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

(a) Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

row-reduce:

$$\begin{bmatrix}1&1&1\\1&2&1\\1&3&2\end{bmatrix} \longrightarrow \begin{bmatrix}1&1&1\\0&1&0\\0&2&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Note: this was self-check problem 2.3.23.

Pivot in each column $\Rightarrow$ sol'ns to $\begin{bmatrix}1&1&1\\1&2&1\\1&3&2\end{bmatrix}\vec{x} = \vec{0}$ have no free variables

$\Rightarrow$ columns are lin. indep.

(b) Find the unique scalars c_1, c_2, c_3 such that the vector

$$\vec{v} = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

is equal to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

same row ops., but w/ the aug. matrix:

$$\left[\begin{array}{ccc|c}1&1&1&2\\1&2&1&1\\1&3&2&3\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&1&1&2\\0&1&0&-1\\0&2&1&1\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c}1&0&1&3\\0&1&0&-1\\0&0&1&3\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&-1\\0&0&1&3\end{array}\right].$$

$$\boxed{c_1 = 0, \quad c_2 = -1, \quad c_3 = 3}$$

3. [9 points] Consider the two bases $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $[I]_B^{B'}$.

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

so $\left[\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right]_{B'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

so $\left[\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$\boxed{[I]_B^{B'} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}}$$

4. [9 points] Let $T: V \to V$ be a linear tranformation.

(a) Prove that $T(\vec{0}) = \vec{0}$.

Here are two possible proofs:

proof 1

$\forall \vec{v} \in V, \quad T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v})$ (linearity)

but $0 \cdot \vec{u} = \vec{0}$ for any vector $\vec{u}$. So this means

$$T(\vec{0}) = \vec{0}.$$

proof 2

Observe that

$$\begin{aligned} T(\vec{0}) &= T(\vec{0}+\vec{0}) \\ &= T(\vec{0}) + T(\vec{0}). \quad \text{(linearity)} \\ \Rightarrow T(\vec{0}) - T(\vec{0}) &= T(\vec{0}) + T(\vec{0}) - T(\vec{0}) \\ \Rightarrow \vec{0} &= T(\vec{0}). \end{aligned}$$

(b) Suppose that $\vec{u}, \vec{v}, \vec{w}$ are three vectors in V, and that the three vectors $T(\vec{u}), T(\vec{v}), T(\vec{w})$ are linearly independent. Prove that the three vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

Suppose that c_1, c_2, c_3 are constants s.t.

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}.$$

We wish to show that $c_1 = c_2 = c_3 = 0$.

Apply T to both sides:

$$T(c_1\vec{u} + c_2\vec{v} + c_3\vec{w}) = T(\vec{0})$$

$$\Rightarrow \underbrace{c_1T(\vec{u}) + c_2T(\vec{v}) + c_3T(\vec{w})}_{\text{(by linearity)}} = \underbrace{\vec{0}}_{\text{(by part a)}}.$$

Since $T(\vec{u}), T(\vec{v}), T(\vec{w})$ are linearly independent, it follows that

$$c_1 = c_2 = c_3 = 0,$$

as desired.

5. [9 points] Consider the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

(a) Find a basis for the column space of A.

row-reduce:

$$\begin{array}{c} \\ - \\ -2\times \\ \\ \end{array}\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{c} - \\ \\ - \\ \\ \end{array}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots in col's 1 & 3

$$\Rightarrow \boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}}$$ is a basis for the column space.

(b) Find a basis for the null space of A.

the sol'n to $A\vec{x} = \vec{0}$ is given by this aug. matrix in RREF:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

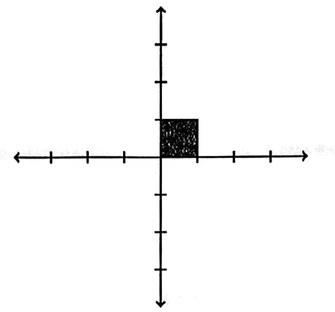
x_2, x_4 free
gen'l sol'n is

$x_1 = -2t + s$
$x_2 = t$
$x_3 = -s$
$x_4 = s$

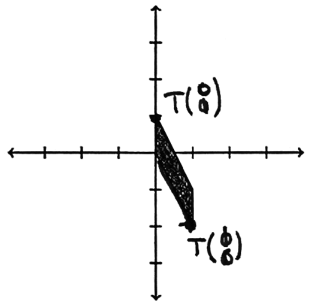
$$\begin{pmatrix} -2t+s \\ t \\ -s \\ s \end{pmatrix} = t \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

so $\boxed{\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$ is a basis for $N(A)$.

6. Consider the square in the plane with vertices $(0,0), (1,0), (0,1), (1,1)$.



(a) [3 points] Find the matrix representation (relative to the standard basis) of a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that transforms the square to the figure shown.

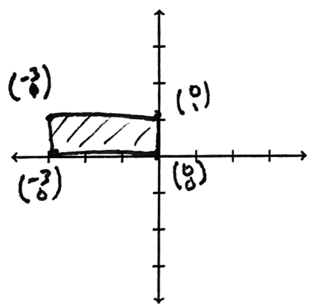


$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$ & $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$A = \boxed{\begin{pmatrix}1 & 0\\-2 & 1\end{pmatrix}}$

(vertical shear)

(b) [3 points] Draw on the axes below the result of transforming the square by the transformation $S: \mathbb{R}^2 \to \mathbb{R}^2$ represented (in the standard basis) by $\begin{bmatrix}-3 & 0\\0 & 1\end{bmatrix}$.



(horiz. dilation)

(c) [3 points] Determine the matrix representation (in the standard basis) for the transformation $S \circ T$ (the composition of S and T, where S is the transformation in (b) and T is the transformation in (a)).

$$\underbrace{\begin{bmatrix}-3 & 0\\0 & 1\end{bmatrix}}_{\substack{\text{trans. in (b)}\\(S)}} \cdot \underbrace{\begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}}_{\substack{\text{trans. in (a)}\\(T)}} = \boxed{\begin{bmatrix}-3 & 0\\-2 & 1\end{bmatrix}}$$