**NOTE (2025): some material on this exam is content we haven't covered yet. See notes in blue about this.**

1. **Short answer questions.** No explanations are necessary.
   (a) [3 points] Give a basis for each of the following vector spaces.
   - $\mathbb{R}^3$
   - $M_{2\times 2}$ (the space of $2 \times 2$ matrices)
   - $\mathcal{P}_2$ (the space of polynomials of degree at most 2)

   (b) [3 points] Suppose that $B$ and $B'$ are two bases for a vector space $V$, and the change of basis matrix is
   $$[I]_B^{B'} = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$
   Let $\vec{v} \in V$ be a vector such that $[\vec{v}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Determine $[\vec{v}]_{B'}$.

   (c) [3 points] **Later material; can omit.**
   Suppose that $T : \mathbb{R}^5 \to \mathbb{R}^3$ is a linear transformation, and that $T$ is surjective (onto). Determine $\dim N(T)$ (where $N(T)$ denotes the null space of $T$).

2. [9 points] Consider the following three vectors.
   $$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$
   (a) Show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.
   (b) Find the unique scalars $c_1, c_2, c_3$ such that the vector
   $$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$
   is equal to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

3. [9 points] Consider the two bases $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $[I]_B^{B'}$.

4. [9 points] **This is about later material, but if you just pretend $T$ is an $n \times n$ matrix instead of a "linear transformation," (b) is a good review problem**
   Let $T : V \to V$ be a linear tranformation.
   (a) Prove that $T(\vec{0}) = \vec{0}$.
   (b) Suppose that $\vec{u}, \vec{v}, \vec{w}$ are three vectors in $V$, and that the three vectors $T(\vec{u}), T(\vec{v}), T(\vec{w})$ are linearly independent. Prove that the three vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

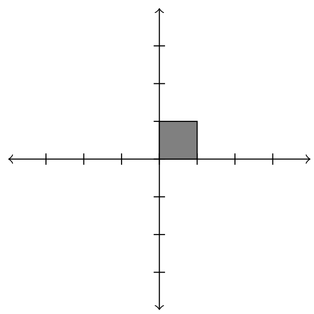
5. [9 points] Consider the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

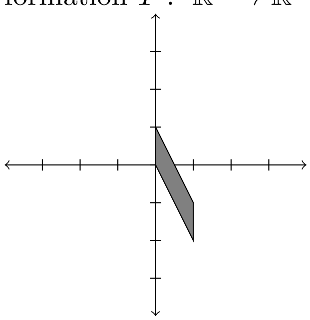
(a) Find a basis for the span of the columns of $A$.

(b) Find a basis for the set of solutions to $A\vec{x} = \vec{0}$.

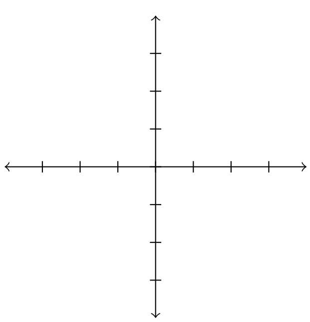
6. Consider the square in the plane with vertices $(0,0), (1,0), (0,1), (1,1)$.



(a) [3 points] Find the matrix representation (relative to the standard basis) of a linear transformation $T\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ that transforms the square to the figure shown.



(b) [3 points] Draw on the axes below the result of transforming the square by the transformation $S\colon\ \mathbb{R}^2 \to \mathbb{R}^2$ represented (in the standard basis) by $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$.



(c) [3 points] Determine the matrix representation (in the standard basis) for the transformation $S \circ T$ (the composition of $S$ and $T$, where $S$ is the transformation in (b) and $T$ is the transformation in (a)).