Amherst College
Department of Mathematics and Statistics

MATH 272 | MIDTERM 2 | SPRING 2018

NAME: Solutions

Read This First!

- You are allowed one page of notes, front and back. No other books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- Please read each question carefully. Show **ALL** work clearly in the space provided. There is an extra page at the back for additional scratchwork.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	6	Total
Points:	9	9	15	9	9	9	60
Score:							

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1. [9 points] **Short answer questions.** No explanations are necessary.

   (a) State the dimension of each of the following vector spaces.

   - $\mathbb{R}^5$ 5
   - $M_{2\times 3}$ 6
   - $\mathcal{P}_3$ 4

   (b) Find the angle between the vectors $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

   dot product: $0+2-2=0$

   $\boxed{90^\circ}$ $(\pi/2)$

   (c) For each of the following subsets of $\mathbb{R}^2$, determine whether or not it is a subspace of $\mathbb{R}^2$. You do not need to prove your answer; simply state "yes" or "no."

   - $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : x+5y=0\right\}$ yes
   - $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : y=x^2\right\}$ no
   - $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : 2x+3y=1\right\}$ no
   - $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : y=7x\right\}$ yes

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2. [9 points] Consider the two bases $B = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $[I]_B^{B'}$.

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ \underset{-5}{5} & \underset{-5/3}{2} & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 1/3 & -1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \end{array}\right)$$

So $\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]_{B'} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

Also, $\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{B'} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by inspection (it matches the second vector of B').

Hence $[I]_B^{B'} = \boxed{\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}}$

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3. [15 points] Consider the following three vectors in $\mathbb{R}^4$.

$$\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

Denote by W the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.

(a) Find a basis of W.

$$\begin{pmatrix} 1 & -2 & 0 \\ -1 & 2 & 3 \\ 1 & -2 & 0 \\ -1 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

pivots in columns 1 & 3 $\Rightarrow$ $\boxed{\{\vec{u}, \vec{w}\}}$ is a basis of W.

(b) What is the dimension of W?

$\boxed{2}$ (the basis has 2 elements).

(Parts (c) and (d) on reverse side)

(c) Find an *orthonormal* basis for W. (Recall that a basis is called *orthonormal* if any two vectors in the basis are orthogonal, and each vector has norm equal to 1.)

First, ~~the~~ replace $\vec{u}, \vec{w}$ by orthogonal vectors:

$$\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\vec{w} - \mathrm{proj}_{\vec{u}}(\vec{w}) = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} - \frac{\vec{u}\cdot\vec{w}}{\vec{u}\cdot\vec{u}}\cdot\vec{u} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} - \frac{-4}{4}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

Then normalize:

$$\frac{1}{\sqrt{1+1+1+1}}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\frac{1}{\sqrt{1+4+1+0}}\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{pmatrix}$$

$$\boxed{\left\{ \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{pmatrix} \right\}}$$

(d) What element of W is closest to the vector $\vec{b} = \begin{pmatrix} 12 \\ 0 \\ 0 \\ 0 \end{pmatrix}$? (In other words, which element $\vec{x}$ of W minimizes $\|\vec{x} - \vec{b}\|$?)

To minimize $\|c_1\vec{u} + c_2\vec{w} - \vec{b}\|$, we can solve the normal eq'n:

$$\begin{pmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{w} \\ \vec{w}\cdot\vec{u} & \vec{w}\cdot\vec{w} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \vec{u}\cdot\vec{b} \\ \vec{w}\cdot\vec{b} \end{pmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ -4 & 10 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 4 & -4 & 12 \\ -4 & 10 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 6 & 12 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

So the closest point is $5\vec{u} + 2\vec{w} = 5\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 1 \\ 5 \\ -3 \end{pmatrix}}$

Another method: add the projections of $\vec{b}$ onto the ortho. vectors from (c).

4. [9 points] Suppose that $\{\vec{u}, \vec{v}\}$ is a basis for a vector space V. Prove that $\{3\vec{u} + 2\vec{v},\ \vec{u} + \vec{v}\}$ is also a basis for V.

Linear independence

Suppose $c_1(3\vec{u}+2\vec{v}) + c_2(\vec{u}+\vec{v}) = \vec{0}$.

Then $(3c_1+c_2)\vec{u} + (2c_1+c_2)\vec{v} = 0$.

Since $\vec{u}, \vec{v}$ are LI, it follows that

$3c_1 + c_2 = 0$ & $2c_1 + c_2 = 0$,

ie. $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Reducing, $\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$,

so the only possible values of c_1, c_2 are $\underline{c_1 = 0}$, $\underline{c_2 = 0}$.

Hence $\{3\vec{u}+2v, \vec{u}+\vec{v}\}$ is LI.

Span

Since $\{3\vec{u}+2\vec{v}, \vec{u}+\vec{v}\}$ is LI, its span is 2-dimensional.

Now, V itself is 2-dim'l (it has a basis of two elements),

so the span of $\{3\vec{u}+2v, \vec{u}+\vec{v}\}$ must be all of V.

Hence $\boxed{\{3\vec{u}+2\vec{v}, \vec{u}+\vec{v}\}}$ is a basis for V.

Note Another way to check that the set spans: observe that

$\vec{u} = 1\cdot(3u+2\vec{v}) - 2(\vec{u}+\vec{v})$

& $\vec{v} = -1(3\vec{u}+2\vec{v}) + 3(\vec{u}+\vec{v})$

Hence $\{\vec{u}, \vec{v}\} \subseteq \operatorname{span}\{3\vec{u}+2\vec{v}, \vec{u}+\vec{v}\}$,

and therefore all of V lies in this span, since $\{\vec{u}, \vec{v}\}$ spans V.

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5. [9 points] Consider the vector space $C[0,1]$ of continuous functions on the interval $[0,1]$. Regard this as an inner product space, with the following inner product.

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

You measure a function $f(x)$ in the laboratory, whose graph appears to be well-approximated by a straight line. Using the experimental data, you compute the following integrals.

$$\begin{aligned}\int_0^1 f(x)\,dx &= 6 &= \langle 1, f\rangle\\ \int_0^1 xf(x)\,dx &= 5 &= \langle x, f\rangle\end{aligned}$$

Determine the constants c_1, c_2 such that $\int_0^1 (c_1 + c_2x - f(x))^2\,dx$ will be minimized (so that $y = c_1 + c_2x$ will approximate $f(x)$ as well as possible, in total-squared-error sense).

It suffices to ensure that

$$1 \perp (c_1 + c_2x - f(x))$$
$$\& \quad x \perp (c_1 + c_2x - f(x))$$

Note 1 & x are not orthogonal, so it is not enough to project onto 1 & x.

This amounts to the linear system:

$$\begin{pmatrix}\langle 1,1\rangle & \langle 1,x\rangle\\ \langle x,1\rangle & \langle x,x\rangle\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}\langle 1,f\rangle\\ \langle x,f\rangle\end{pmatrix}$$

$\langle 1,1\rangle = \int_0^1 1\,dx = 1$; $\quad \langle 1,x\rangle = \int_0^1 x\,dx = \frac{1}{2}$; $\quad \langle x,x\rangle = \int_0^1 x^2\,dx = 1/3$

so the system is

$$\begin{pmatrix}1 & 1/2\\ 1/2 & 1/3\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}6\\ 5\end{pmatrix}$$

$$\left(\begin{array}{cc|c}1 & 1/2 & 6\\ 1/2 & 1/3 & 5\end{array}\right) \begin{array}{ccc}-1/2 & -1/4 & -3\end{array} \longrightarrow \left(\begin{array}{cc|c}1 & 1/2 & 6\\ 0 & 1/12 & 2\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}1 & 0 & -6\\ 0 & 1 & 24\end{array}\right)$$

& the best constants are $\boxed{c_1 = -6 \quad c_2 = 24}$.

(the approximating function is $\underline{-6+24x}$)

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6. [9 points] Suppose that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is used to draw a three-dimensional scene on a page. A point $\vec{x} \in \mathbb{R}^3$ will be drawn at the location $T(\vec{x})$ on the page (which is identified with $\mathbb{R}^2$ by some choice of coordinates).

Suppose that the following three equations hold.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-4/5\\3/5\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

In other words, the location $\begin{bmatrix}1\\0\\0\end{bmatrix}$ in the scene is drawn at location $\begin{bmatrix}1\\0\end{bmatrix}$ on the page, the location $\begin{bmatrix}0\\1\\0\end{bmatrix}$ in the scene is drawn at location $\begin{bmatrix}-4/5\\3/5\end{bmatrix}$ on the page, and the location $\begin{bmatrix}0\\0\\1\end{bmatrix}$ in the scene is drawn at location $\begin{bmatrix}0\\1\end{bmatrix}$ on the page.

(a) Suppose that one corner of a structure in the scene is located at the coordinates $\begin{bmatrix}3\\5\\1\end{bmatrix}$.

Where on the page should it be drawn?

$$T\left(\begin{pmatrix}3\\5\\1\end{pmatrix}\right) = 3T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) + 5T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) + 1\cdot T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= 3\begin{pmatrix}1\\0\end{pmatrix} + 5\begin{pmatrix}-4/5\\3/5\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}3-4+0\\0+3+1\end{pmatrix} = \boxed{\begin{pmatrix}-1\\4\end{pmatrix}}$$

(b) Find a matrix A such that $T(\vec{v}) = A\vec{v}$ (this matrix A is called the matrix representation of T).

$$A = \boxed{\begin{pmatrix}1 & -4/5 & 0\\0 & 3/5 & 1\end{pmatrix}}$$

$$\begin{array}{ccc}\uparrow & \uparrow & \uparrow\\ \text{image of } \vec{e}_1 & \text{image of } \vec{e}_2 & \text{image of } \vec{e}_3\end{array}$$

(part (c) on reverse side)

For convenience, the three equations from the previous page are reproduced here:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-4/5\\3/5\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}$$

(c) Determine a basis for the nullspace $N(T)$ of T. What is its dimension?

$$\begin{pmatrix} 1 & -4/5 & 0 \\ 0 & 3/5 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 5/3 \end{pmatrix}$$

so gen'l sol'n to $A\vec{x}=0$ is $x_3 \cdot \begin{pmatrix} -4/3 \\ -5/3 \\ 1 \end{pmatrix}$ (x_3 free)

so $\boxed{\left\{ \begin{pmatrix} -4/3 \\ -5/3 \\ 1 \end{pmatrix} \right\}}$ is a basis for $N(T)$,

which is $\boxed{\text{1-dimensional.}}$.

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