NOTE (2025): some material on this exam is content we haven't covered yet. See notes in blue about this.

1. [9 points] **Short answer questions.** No explanations are necessary.
   (a) State the dimension of each of the following vector spaces.
   - $\mathbb{R}^5$
   - $M_{2\times 3}$
   - $\mathcal{P}_3$

   (b) Find the angle between the vectors $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

   (c) For each of the following subsets of $\mathbb{R}^2$, determine whether or not it is a subspace of $\mathbb{R}^2$. You do not need to prove your answer; simply state "yes" or "no."
   - $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x + 5y = 0 \right\}$
   - $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = x^2 \right\}$
   - $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 2x + 3y = 1 \right\}$
   - $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 7x \right\}$

2. [9 points] Consider the two bases $B = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. Find the change of basis matrix $[I]_B^{B'}$.

3. [15 points] Consider the following three vectors in $\mathbb{R}^4$.

$$\vec{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

   Denote by W the span of $\{\vec{u}, \vec{v}, \vec{w}\}$.

   (a) Find a basis of W.

   (b) What is the dimension of W? *(Parts (c) and (d) on reverse side)*

   (c) **Later material; can omit.**
   Find an *orthonormal* basis for W. (Recall that a basis is called *orthonormal* if any two vectors in the basis are orthogonal, and each vector has norm equal to 1.)

   (d) What element of W is closest to the vector $\vec{b} = \begin{pmatrix} 12 \\ 0 \\ 0 \\ 0 \end{pmatrix}$? (In other words, which element $\vec{x}$ of W minimizes $\|\vec{x} - \vec{b}\|$?)

4. [9 points] Suppose that $\{\vec{u}, \vec{v}\}$ is a basis for a vector space V. Prove that $\{3\vec{u} + 2\vec{v},\ \vec{u} + \vec{v}\}$ is also a basis for V.

5. [9 points] **Later material; can omit.**
Consider the vector space $\mathcal{C}[0, 1]$ of continuous functions on the interval $[0, 1]$. Regard this as an inner product space, with the following inner product.

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\ dx$$

You measure a function $f(x)$ in the laboratory, whose graph appears to be well-approximated by a straight line. Using the experimental data, you compute the following integrals.

$$\begin{aligned} \int_0^1 f(x)\ dx &= 6 \\ \int_0^1 xf(x)\ dx &= 5 \end{aligned}$$

Determine the constants c_1, c_2 such that $\int_0^1 (c_1 + c_2x - f(x))^2\ dx$ will be minimized (so that $y = c_1 + c_2x$ will approximate $f(x)$ as well as possible, in total-squared-error sense).

6. [9 points] **Later material; can omit.**
Suppose that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is used to draw a three-dimensional scene on a page. A point $\vec{x} \in \mathbb{R}^3$ will be drawn at the location $T(\vec{x})$ on the page (which is identified with $\mathbb{R}^2$ by some choice of coordinates).

Suppose that the following three equations hold.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-4/5\\3/5\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

In other words, the location $\begin{bmatrix}1\\0\\0\end{bmatrix}$ in the scene is drawn at location $\begin{bmatrix}1\\0\end{bmatrix}$ on the page, the location $\begin{bmatrix}0\\1\\0\end{bmatrix}$ in the scene is drawn at location $\begin{bmatrix}-4/5\\3/5\end{bmatrix}$ on the page, and the location $\begin{bmatrix}0\\0\\1\end{bmatrix}$ in the scene is drawn at location $\begin{bmatrix}0\\1\end{bmatrix}$ on the page.

(a) Suppose that one corner of a structure in the scene is located at the coordinates $\begin{bmatrix}3\\5\\1\end{bmatrix}$.
Where on the page should it be drawn?

(b) Find a matrix A such that $T(\vec{v}) = A\vec{v}$ (this matrix A is called the matrix representation of T).

(part (c) on reverse side)

For convenience, the three equations from the previous page are reproduced here:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}\qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-4/5\\3/5\end{bmatrix}\qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}$$

(c) Determine a basis for the nullspace $N(T)$ of T. What is its dimension?