NOTE (2025): some material on this exam is content we haven't covered yet. See notes in blue about this.

1. [9 points] **Short answer questions.** No explanations are necessary.

   (a) The set $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$. Compute the coordinates $\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} \right]_B$.

   (b) **Later material; can omit.**
   Give an example of an inner product on the vector space $\mathcal{C}[-\pi, \pi]$ of continuous functions on $[-\pi, \pi]$, by writing a formula for the inner product of two functions $f(x)$ and $g(x)$ below.

   $$\langle f(x), g(x) \rangle =$$

   (c) Find the scalar multiple of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ that is closest to $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$ (in the sense of minimizing the norm of the difference).

2. [9 points] Consider the following two bases for $\mathbb{R}^2$ (you may assume that these are indeed bases).

$$\begin{aligned} B &= \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\} \\ B' &= \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\} \end{aligned}$$

   Determine the transition matrix $[I]_B^{B'}$.

3. [9 points] Let W be the following subset of $\mathbb{R}^3$.

$$W = \left\{ \begin{pmatrix} r+s \\ r+t \\ s-t \end{pmatrix} : r, s, t \in \mathbb{R} \right\}$$

   (a) Show that W is a subspace of $\mathbb{R}^3$ (in class I referred to sets like this as "parameterized subspaces").

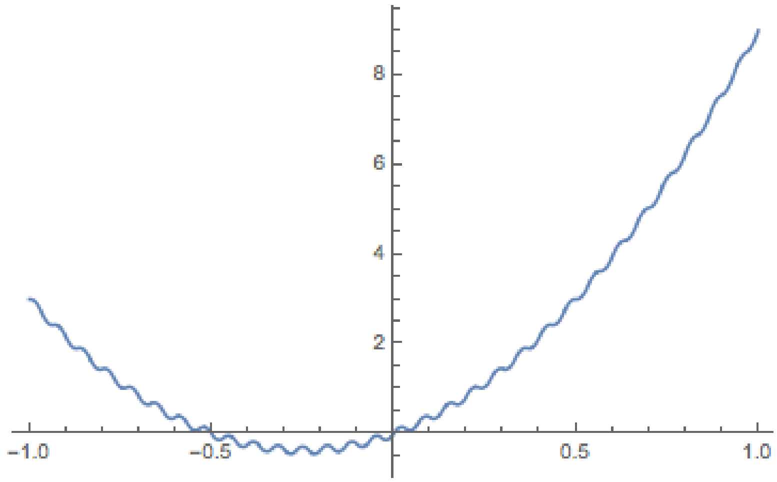
   (b) Find a basis for W, and determine $\dim W$.

4. [9 points] Suppose that $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent set of vectors in a vector space V, and that $c_1, c_2, c_3, d_1, d_2, d_3$ are scalars such that

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = d_1\vec{u} + d_2\vec{v} + d_3\vec{w}.$$

   Prove that $c_1 = d_1, c_2 = d_2$, and $c_3 = d_3$.

5. [9 points] **Later material; can omit.**
   A signal is measured in a lab, giving a function $f(x)$ on the interval $[-1, 1]$ shown below. Based on the graph, you guess that this signal can be approximated by a linear combination of x and x^2.



In order to produce a good model, you numerically compute the following integrals.

$$\int_{-1}^{1} x \cdot f(x)\, dx = 2.00$$
$$\int_{-1}^{1} x^2 \cdot f(x)\, dx = 2.40$$

Regard $f(x), x$, and x^2 as elements of the inner product space $\mathcal{C}[-1,1]$, with the usual inner product.

(a) Compute the projections $\text{proj}_x f(x)$ and $\text{proj}_{x^2} f(x)$.

(b) Show that $x \perp x^2$ in this inner product space.

(c) Find coefficients a_1, a_2 minimizing the following objective function.

$$\int_{-1}^{1} \left(a_1 x + a_2 x^2 - f(x)\right)^2\, dx.$$

The function $g(x) = a_1 x + a_2 x^2$ will be a quadratic function approximating $f(x)$ as well as possible in the "least squares" sense.