Amherst College
Department of Mathematics and Statistics

MATH 272 MIDTERM 2 SPRING 2019

NAME: Solutions

Read This First!

- Keep cell phones off and out of sight.
- Do not talk during the exam.
- You are allowed one page of notes, front and back.
- No calculators or other devices are permitted.
- You may use any of the blank pages to continue answers if you run out of space. Please clearly indicate on the problem's original page if you do so, so that I know to look for it.
- In order to receive full credit on a problem, solution methods must be complete, logical and understandable.

Grading - For Instructor Use Only

Question:	1	2	3	4	5	Total
Points:	9	9	9	9	9	45
Score:						

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1. [9 points] **Short answer questions** (no explanation or shown work is necessary for these questions)

   (a) Suppose that $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$, such that the following three inner products hold.

   $$\vec{u} \cdot \vec{u} = 7, \qquad \vec{u} \cdot \vec{v} = 2, \qquad \vec{v} \cdot \vec{v} = 5.$$

   Determine the norm $\|\vec{u} + \vec{v}\|$.

   $$\|\vec{u}+\vec{v}\| = \sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})}$$
   $$= \sqrt{\vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}}$$
   $$= \sqrt{7+2\cdot 2+5}$$
   $$= \sqrt{16}$$
   $$= \boxed{4}$$

   (b) Suppose that $B = \left\{ \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\}$. This is a basis of $\mathbb{R}^2$. Let S denote the standard basis of $\mathbb{R}^2$. What is the change of basis matrix $[I]_B^S$?

   $$[I]_B^S = \boxed{\begin{pmatrix} 5 & 3 \\ 1 & 7 \end{pmatrix}}$$

   (c) Consider the following basis for $\mathcal{P}_2$: $B = \{x+1, x-1, x^2\}$. Determine the coordinate vector $[x^2+x+1]_B$.

   $$x^2+x+1 = 1\cdot(x+1) + 0\cdot(x-1) + 1\cdot x^2$$
   $$\Rightarrow [x^2+x+1]_B = \boxed{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}.$$

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2. [9 points] Consider the following two bases of $\mathbb{R}^3$ (you do not need to prove that these are bases).

$$B = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

$$B' = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}, \begin{pmatrix}1\\5\\6\end{pmatrix} \right\}$$

Determine the change of basis matrix $[I]_B^{B'}$.

$$[I]_B^S = \begin{pmatrix}2 & 1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

$$[I]_{B'}^S = \begin{pmatrix}1 & 0 & 1\\1 & 5 & 5\\0 & 7 & 6\end{pmatrix} \Rightarrow [I]_S^{B'} = \begin{pmatrix}1 & 0 & 1\\1 & 5 & 5\\0 & 7 & 6\end{pmatrix}^{-1}.$$

We can compute this inverse as follows:

$$\left(\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\1_{-1} & 5 & 5_{-1} & 0_{-1} & 1 & 0\\0 & 7 & 6 & 0 & 0 & 1\end{array}\right)$$

$$\xrightarrow{R2 -= R1} \left(\begin{array}{ccc|ccc}1 & 0 & 1 & 1 & 0 & 0\\0 & 5_{/5} & 4_{/5} & -1_{/5} & 1_{/5} & 0\\0 & 7_{-7} & 6_{-28/5} & 0_{+7/5} & 0_{-7/5} & 1\end{array}\right)$$

$$\xrightarrow[R3 -= 7R2]{R2 /= 5} \left(\begin{array}{ccc|ccc}1 & 0 & 1_{-1} & 1_{-7/2} & 0_{+7/2} & 0_{-5/2}\\0 & 1 & 4/5_{-4/5} & -1/5_{-14/5} & 1/5_{+14/5} & 0_{-2}\\0 & 0 & 2/5_{\times 5/2} & 7/5_{\times 5/2} & -7/5_{\times 5/2} & 1_{\times 5/2}\end{array}\right)$$

$$\xrightarrow[\substack{R2 -= 4/5 R3\\ R1 -= R3}]{R3 \times= 5/2} \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -5/2 & 7/2 & -5/2\\0 & 1 & 0 & -3 & 3 & -2\\0 & 0 & 1 & 7/2 & -7/2 & 5/2\end{array}\right)$$

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hence $$[I]_S^{B'} = \begin{pmatrix} -5/2 & 7/2 & -5/2 \\ -3 & 3 & -2 \\ 7/2 & -7/2 & 5/2 \end{pmatrix}$$

and

$$[I]_B^{B'} = [I]_S^{B'} \cdot [I]_B^S$$

$$= \begin{pmatrix} -5/2 & 7/2 & -5/2 \\ -3 & 3 & -2 \\ 7/2 & -7/2 & 5/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -3/2 & 1 & -5/2 \\ -3 & 0 & -2 \\ 7/2 & 0 & 5/2 \end{pmatrix}}$$

Alt. solution:

row-reduce the matrix $\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & 5 & 5 & 1 & 1 & 0 \\ 0 & 7 & 7 & 0 & 0 & 1 \end{array}\right)$ with $\underbrace{}_{B' \text{ vectors}}$ $\underbrace{}_{B \text{ vectors}}$

(in effect, this computes $\left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right]_{B'}$, $\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right]_{B'}$, & $\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]_{B'}$

all at once).

to obtain $$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & 1 & -5/2 \\ 0 & 1 & 0 & -3 & 0 & -2 \\ 0 & 0 & 1 & 7/2 & 0 & 5/2 \end{array}\right)$$ with $\underbrace{}_{[I]_B^{B'}}$

3. [9 points] Let W denote the set of solutions $\vec{x} \in \mathbb{R}^4$ to the following matrix equation.

$$\begin{pmatrix}1&0&0&1\\2&3&9&5\end{pmatrix}\vec{x} = \vec{0}$$

(a) Prove that W is a *subspace* of $\mathbb{R}^4$.

It suffices to verify that $\{$W is nonempty, and $\forall \vec{x},\vec{y} \in W$, $\forall c \in \mathbb{R}$, $\vec{x}+c\vec{y} \in W$.

$\Big\{$ W is nonempty since $\begin{pmatrix}1&0&0&1\\2&3&9&5\end{pmatrix}\cdot\vec{0}=\vec{0}$, so $\vec{0} \in W$.

So suppose that * $\begin{pmatrix}1&0&0&1\\2&3&9&5\end{pmatrix}\vec{x}=\vec{0}$ & $\begin{pmatrix}1&0&0&1\\2&3&9&5\end{pmatrix}\vec{y}=\vec{0}$

(ie. $\vec{x},\vec{y} \in W$).

Then

$$\begin{pmatrix}1&0&0&1\\2&3&9&5\end{pmatrix}(\vec{x}+c\vec{y}) = \begin{pmatrix}1&0&0&1\\2&3&9&5\end{pmatrix}\vec{x} + c\begin{pmatrix}1&0&0&1\\2&3&9&5\end{pmatrix}\vec{y}$$
$$= \vec{0}+c\cdot\vec{0}$$
$$=\vec{0},$$

hence $\vec{x}+c\vec{y} \in W$ as well.

Thus W is a subspace of $\mathbb{R}^4$.

(b) Find a *basis* for W.

compare this to prob. 3.3.40 (on PSet 8)

$$\begin{pmatrix}1&0&0&1\\2_{-2}&3&9&5_{-2}\end{pmatrix} \xrightarrow{R2-=2R1} \begin{pmatrix}1&0&0&1\\0&3_{/3}&9_{/3}&3_{/3}\end{pmatrix}$$
$$\xrightarrow{R2/=3} \begin{pmatrix}1&0&0&1\\0&1&3&1\end{pmatrix}$$

$\Rightarrow$ gen'l sol'n to $\begin{pmatrix}1&0&0&1\\2&3&9&5\end{pmatrix}\vec{x}=\vec{0}$ is

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-x_4\\-3x_3-x_4\\x_3\\x_4\end{pmatrix} = x_3\begin{pmatrix}0\\-3\\1\\0\end{pmatrix} + x_4\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix} \quad (x_3, x_4 \text{ free})$$

$\Rightarrow$ $\boxed{\left\{\begin{pmatrix}0\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}\right\}}$ is a basis for W.

(continued on reverse)

(c) Determine the *dimension* of W.

$\dim W = 2$ (basis has two elements).

(d) Find the vector $\vec{w}$ in W that is closest to the vector $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 4 \\ -1 \end{pmatrix}$.

Let $A = \begin{pmatrix} 0 & -1 \\ -3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ (columns are basis of W).

The normal eq'n is:

$$A^t A \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^t \begin{pmatrix} 2 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

ie. $$\begin{pmatrix} 0 & -3 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{10 \cdot 3 - 3 \cdot 3} \begin{pmatrix} 3 & -3 \\ -3 & 10 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$= \frac{1}{21} \begin{pmatrix} 21 \\ -42 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

so the nearest LC of the basis vectors is

$$1 \cdot \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}}.$$

4. [9 points] Suppose that A is an invertible $n \times n$ matrix, and $\{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent set in $\mathbb{R}^n$. Prove that $\{A\vec{u}, A\vec{v}, A\vec{w}\}$ is also a linearly independent set.

Suppose that $a, b, c \in \mathbb{R}$ satisfy

~~$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}.$~~

$$a(A\vec{u}) + b \cdot (A\vec{v}) + c(A\vec{w}) = \vec{0}.$$

Then

$$A \cdot [a\vec{u} + b\vec{v} + c\vec{w}] = \vec{0}.$$

Multiplying by A^{-1}, this implies

$$I \cdot [a\vec{u} + b\vec{v} + c\vec{w}] = \vec{0},$$

ie.

$$a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}.$$

Since $\underline{\{\vec{u}, \vec{v}, \vec{w}\} \text{ is LI}}$, it follows that

$$\underline{a = b = c = 0}.$$

Thus no nontrivial LC of $\{A\vec{u}, A\vec{v}, A\vec{w}\}$ is $\vec{0}$,

ie. $\{A\vec{u}, A\vec{v}, A\vec{w}\}$ is LI.

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5. [9 points] Consider the vector space $C[-1,1]$ of continuous functions on $[-1,1]$, equipped with the following inner product.

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

(a) Show that, under this inner product, $1 \perp x$ (here 1 denotes the constant function $f(x) = 1$, while x denotes the function $g(x) = x$).

$$\langle 1, x\rangle = \int_{-1}^{1} 1 \cdot x \, dx$$

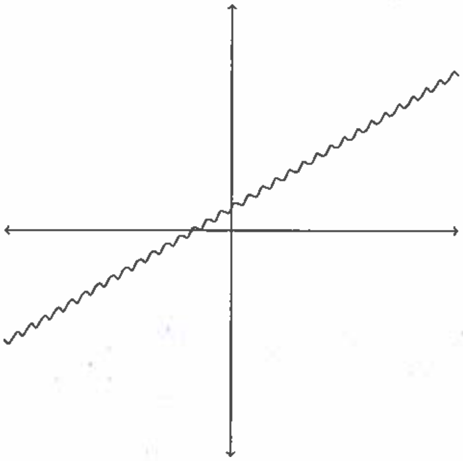
$$= \int_{-1}^{1} x \, dx = \left[\tfrac{1}{2}x^2\right]_{-1}^{1}$$

$$= \tfrac{1}{2}(1^2 - (-1)^2) = \underline{0}.$$

Hence $1 \perp x$.

(continued on reverse)

(b) In the laboratory, you measure a function $f(x)$ on $[-1, 1]$, whose graph is shown below.



Based on the graph, you suspect that this function is approximately equal to a function of the form $g(x) = c_1x + c_2$. In order to find a good fit, you compute the following two integrals using your numerical data.

$$\int_{-1}^{1} f(x)\,dx = 0.2$$

$$\int_{-1}^{1} xf(x)\,dx = 0.4$$

Using these computations, compute the following two projections (in the inner product space described above):

$$\text{proj}_1 f(x), \quad \text{proj}_x f(x).$$

$$\text{proj}_1 f(x) = \frac{\langle 1, f(x)\rangle}{\langle 1, 1\rangle}\cdot 1 = \frac{\int_{-1}^{1} f(x)dx}{\int_{-1}^{1} 1dx}1 = \frac{\overset{\text{given}}{0.2}}{2}\cdot 1$$

$$= \boxed{0.1 \cdot 1}$$

$$\text{proj}_x f(x) = \frac{\langle x, f(x)\rangle}{\langle x, x\rangle}\cdot x = \frac{\int_{-1}^{1} xf(x)dx}{\int_{-1}^{1} x^2dx}\cdot x$$

$$= \frac{\overset{\text{given}}{0.4}}{\left[\frac{1}{3}x^3\right]_{-1}^{1}}\cdot x = \frac{0.4}{2/3}x = \frac{3\cdot 0.4}{2}\cdot x$$

$$= \boxed{0.6\,x}$$

(indeed, $y = 0.1 + 0.6x$ is a good approximation of the graph above).