NOTE (2025): some material on this exam is content we haven't covered yet. See notes in blue about this.

1. [9 points] **Short answer questions** (no explanation or shown work is necessary for these questions)

   (a) Suppose that $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$, such that the following three inner products hold.
   $$\vec{u} \cdot \vec{u} = 7, \qquad \vec{u} \cdot \vec{v} = 2, \qquad \vec{v} \cdot \vec{v} = 5.$$
   Determine the norm $\|\vec{u} + \vec{v}\|$.

   (b) Suppose that $B = \left\{ \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\}$. This is a basis of $\mathbb{R}^2$. Let S denote the standard basis of $\mathbb{R}^2$. What is the change of basis matrix $[I]_B^S$?

   (c) Consider the following basis for $\mathcal{P}_2$: $B = \{x+1, x-1, x^2\}$. Determine the coordinate vector $[x^2 + x + 1]_B$.

2. [9 points] Consider the following two bases of $\mathbb{R}^3$ (you do not need to prove that these are bases).
   $$B = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$
   $$B' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} \right\}$$
   Determine the change of basis matrix $[I]_B^{B'}$.

3. [9 points] Let W denote the set of solutions $\vec{x} \in \mathbb{R}^4$ to the following matrix equation.
   $$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 3 & 9 & 5 \end{pmatrix} \vec{x} = \vec{0}$$

   (a) Prove that W is a *subspace* of $\mathbb{R}^4$.

   (b) Find a *basis* for W.

   (c) Determine the *dimension* of W.

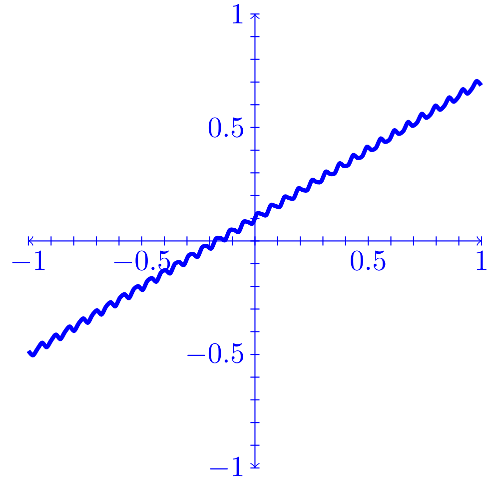
   (d) Find the vector $\vec{w}$ in W that is closest to the vector $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 4 \\ -1 \end{pmatrix}$.

4. [9 points] Suppose that A is an invertible $n \times n$ matrix, and $\{\vec{u}, \vec{v}, \vec{w}\}$ is a linearly independent set in $\mathbb{R}^n$. Prove that $\{A\vec{u}, A\vec{v}, A\vec{w}\}$ is also a linearly independent set.

5. [9 points] **Later material; can omit.**
   Consider the vector space $\mathcal{C}[-1,1]$ of continuous functions on $[-1,1]$, equipped with the following inner product.
   $$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx$$

(a) Show that, under this inner product, $1 \perp x$ (here 1 denotes the constant function $f(x) = 1$, while x denotes the function $g(x) = x$).

(b) In the laboratory, you measure a function $f(x)$ on $[-1, 1]$, whose graph is shown below.



Based on the graph, you suspect that this function is approximately equal to a function of the form $g(x) = c_1 x + c_2$. In order to find a good fit, you compute the following two integrals using your numerical data.

$$\int_{-1}^{1} f(x)\ dx = 0.2 \qquad\qquad \int_{-1}^{1} x f(x)\ dx = 0.4$$

Using these computations, compute the following two projections (in the inner product space described above):

$$\mathrm{proj}_1 f(x), \quad \mathrm{proj}_x f(x).$$

(As we discussed in class, the fact that $1 \perp x$ means that the sum $\mathrm{proj}_1 f(x), +\mathrm{proj}_x f(x)$ is the linear combination of 1 and x that best approximates $f(x)$. You can use this, plus the graph, to check your answers.)