Study guide

- (§4.1) Know the definition of a "linear transformation," and their basic properties.
- (§4.1) Be comfortable with the notation $T:\ V \to W$.
- (§4.1) Understand why linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ is always represetented by an $m \times n$ matrix (and what "represented by" means in this context).
- (§4.2) Given a matrix A, know the definitions of $N(A)$ and $R(A)$, and how to compute bases for both of them (both are reformulations of algorithms studied earlier in the course).
- (§4.2) Know the "rank-nullity theorem." (Theorem 5 in the textbook)
- (§4.4) Know the definition of $[T]_B^{B'}$.
- (§4.4) Know how to compute the matrix $[T]_B^{B'}$ one column at a time.
- (§5.1) Know the terms: eigenvector, eigenvalue, characteristic equation.
- (§5.2) Know how to find the eigenvalues of a matrix, and how to find the eigenspace for each eigenvalue.
- (§5.2) Know how to diagonalize a matrix.

1. (Textbook 4.1.30)
   Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ 3x + y \\ 2y \end{bmatrix}$$

   (a) Find a matrix A such that $T(\mathbf{v}) = A\mathbf{v}$.

   (b) Find $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.

2. (Textbook §4.2, problems 21 through 24: finding a basis for some null spaces)

Note Several problems below refer to "linear operators," which is a term we haven't used in class yet. A "linear operator" is the same thing as a "linear transformation" $T : V \to V$, i.e. a linear transformation from a vector space to itself. A useful bit of notation: for a linear operator we can simply write $[T]_B$ instead of $[T]_B^B$ for a matrix representation using the same basis for both input and output.

3. (Textbook 4.2.32)
   Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for $\mathbb{R}^3$. Suppose

$$T(\mathbf{v}_1) = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad T(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \quad T(\mathbf{v}_3) = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

   (a) Determine whether

$$\mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

is in the range of T.

(b) Find a basis for $R(T)$.

(c) Find $\dim(N(T))$.

4. (Textbook 4.2.40)
Suppose that $T : V \to V$ is a linear operator. Prove if $R(T) = N(T)$, then $\dim V$ is even.

5. (Textbook 4.4.13)
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ x - y \end{bmatrix}$$

Let B be the standard ordered basis for $\mathbb{R}^2$ and B' the ordered basis for $\mathbb{R}^2$ defined by

$$B' = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix} \right\}$$

a) Find $[T]_B$.

b) Find $[T]_{B'}$.

c) Find $[T]_B^{B'}$.

d) Find $[T]_{B'}^{B}$.

e) Let C be the ordered basis obtained by switching the order of the vectors in B. Find $[T]_C^{B'}$.

f) Let C' be the ordered basis obtained by switching the order of the vectors in B'. Find $[T]_{C'}^{B'}$.

Note The remaining problems concern *eigenvectors* and *diagonalization*, which we will cover later in the week.

6. (Textbook §5.2, problems 19-26: diagonalize each matrix.)
You can check the odd-numbered problems in the back of the book.

7. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across the line $y = 7x$.

(a) Find a nonzero vector $\vec{u}$ such that $T(\vec{u}) = \vec{u}$ (i.e. an eigenvector for eigenvalue $\lambda = 1$). You can do this without any computation; think geometrically about T.

(b) Find a nonzero vector $\vec{v}$ such that $T(\vec{v}) = -\vec{v}$ (in other words, an eigenvector for eigenvalue $\lambda = -1$). Again, you can do this without much computation; it's useful to think about the dot product with $\vec{u}$.

(c) Let $B = \{\vec{u}, \vec{v}\}$, where these are the vectors you found in parts (a), (b). This is a basis for $\mathbb{R}^2$ (you don't need to prove this). Find $[T]_B$ (this should not require any computations at all; just use the equations $T(\vec{u}) = \vec{u}$ and $T(\vec{v}) = -\vec{v}$).

(d) Compute $[I]_B^S$, and use it to compute $[T]_S$ (here, S is the standard basis for $\mathbb{R}^2$).

8. In a certain city, researchers aim to model the progression of flu infection. The current state of the infection is recorded in a vector $\begin{pmatrix} h \\ s \end{pmatrix}$, where h denotes the number of healthy people (in millions), and s denotes the number of sick people (in millions). The progression of the disease week-to-week can be modeled by the following equation. Here h', s' denote the number (in millions) of healthy and sick people one week later.

$$\begin{pmatrix} h' \\ s' \end{pmatrix} = \begin{pmatrix} 0.9 & 0.6 \\ 0.1 & 0.4 \end{pmatrix} \begin{pmatrix} h \\ s \end{pmatrix}$$

   (a) According to this model, what percentage of healthy people become sick in the following week? What percentage of sick people become healthy?

   (b) Determine the eigenvalues of the matrix in the model above, and find a corresponding eigenvector for each eigenvalue.

   (c) Suppose that initially, the city has 14 million healthy people and 0 sick people. Express this state as a vector, and write it as a linear combination of the eigenvectors you found in the previous part.

   (d) Using your answer in the previous part, obtain a formula for the state (number of healthy people and number of sick people) after n weeks. Your formula should be expressed in terms of nth powers of the eigenvalues you found in (b).

   (e) What does your answer in (d) suggest about the long-term state of the disease? In other words, after many weeks have passed, roughly how many people do you expect to be sick in the city?

> **Note** This problem touches on several ideas about "Markov Chains," which are discussed in detail in §5.4 of our text. You may find it useful to read that section to think about this problem, but it is not necessary to do so (and I will not expect you to know any vocabulary or theorems from that section).