Study guide

- (§1.5) Terminology: *homogeneous* linear system, *trivial solution.*
- (§1.5) Once you know that $A\vec{x} = \vec{b}$ has one solution, you can find all of the other solutions by solving the homogeneous equation $A\vec{x} = \vec{0}$. Understand why this is.
- (§1.6) Determinants, interpreted as area/volume expansion factor.
- (§1.6) Techniques to evaluate determinants: formula for 2×2 case; row-reduction; cofactor expansion.

1. (Textbook 1.5.26)
   Let $A = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -4 & 8 \\ 3 & -6 & 12 \end{bmatrix}$ Find a nontrivial solution to $A\vec{x} = \vec{0}$. (Terminology note: a "nontrivial solution" to a homogeneous matrix equation is the same as a *nonzero* solution.)

2. (Textbook 1.5.28)
   Find a nonzero 3×3 matrix A such that the vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is a solution to $A\vec{x} = \vec{0}$.

♣ 3. This problem is meant to illustrate why, in the definition of *inverse matrix*, we include both conditions $AA^{-1} = I$ *and* $A^{-1}A = I$. Namely, it is possible to have one but not the other.

   (a) (This part is not a proof; it's enough to find an example and just write it down) Find a 2×3 matrix A and a 3×2 matrix B such that $AB = I$. Check that $BA \neq I$. (How did I know this is true, no matter what you chose? I am not a mind-reader, but the answer will have to wait a bit.)

   (b) Suppose A is an $m \times n$ matrix and B is an $n \times m$ matrix such that $AB = I$ (your example from part (a) is one such pair, but your proof should work for *any* such pair). Prove that for any matrix equation $A\vec{x} = \vec{b}$, the vector $\vec{x} = B\vec{b}$ is *a solution.* That is, we have this implication:
   $$A\vec{x} = \vec{b} \quad \Leftarrow \quad \vec{x} = B\vec{b}.$$
   (Recall that B were an *inverse*, we'd also have the converse implication, as we've seen in class.)

   (c) Now consider again the *specific* matrices A, B you found in part (a), and choose any vector $\vec{b} \in \mathbb{R}^2$ you like. Show that although $B\vec{b}$ is *a solution* to $A\vec{x} = \vec{b}$ (thanks to part (b)), it is *not the only solution.* So the converse implication in part (b) is *false.*

   (d) Now suppose that A is a $m \times n$ matrix and B is an $n \times m$ matrix, and assume that $BA = I$ (note that this is the opposite multiplication order to the one we've been talking about so far!). Prove that for any matrix equation $A\vec{x} = \vec{b}$, the vector $\vec{x} = B\vec{b}$ is *the only possible solution.* That is, we have this implication:
   $$A\vec{x} = \vec{b} \quad \Rightarrow \quad \vec{x} = B\vec{b}.$$
   (Again, though, the converse is false: it's possible that there is *no solution.* Make sure you understand why this doesn't contradict what you just proved; it is a subtle logical point! I encourage you to construct a specific example to see this, but you do not need to write it down to submit with this assignment.)

♣ 4. Let A be an $n \times n$ matrix, and let $B = A - 7I$ (where I is the $n \times n$ identity matrix). Prove that the following two sets are equal:

$$\{\vec{v} \in \mathbb{R}^n : \ A\vec{v} = 7\vec{v}\} = \{\vec{v} \in \mathbb{R}^n : \ B\vec{v} = \vec{0}\}.$$

(Please ask me or one of the course staff about any of this notation if it is new to you! We've used it a bit in class, but it takes some getting used to when you are new to it.)

5. (Textbook 1.6.10)

Answer the following questions about the matrix $A = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix}$

(a) Find the determinant of the matrix by using an expansion along row 4.

(b) Find the determinant of the matrix by using an expansion along row 3.

(c) Find the determinant of the matrix by using an expansion along column 2.

(d) In (a), (b), and (c), which computation do you prefer, and why?

(e) Does the matrix A have an inverse? Do not try to compute the inverse.

6. (Textbook 1.6.22 and 1.6.26)
Determine the following determinants. Decide whether the matrix is invertible, without attempting to compute its inverse.

a) $\det \begin{bmatrix} 1 & 2 & 4 \\ 4 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$

b) $\det \begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & -1 & 0 \end{bmatrix}$