Study guide

- (online notes) Understand the proof that solutions to the normal equation are necessarily solutions to the least-squares problem.
- (§3.1) Understand the notion of a "vector space" intuitively. For this course, you do not need to know or work with the formal axioms.
- (§3.1) Know the notation for the four main vector spaces we work with: $\mathbb{R}^n, \mathcal{P}_d, \mathcal{C}[a,b], M_{m\times n}$ (note: my notation $\mathcal{C}[a,b]$ from class is slightly different from the book's notation; you can feel free to use whichever you like).
- (§3.2) Know the definition of a subspace, and how to formally prove that something is a subspace of a vector space V.
- (§3.2) Know what subsapces of $\mathbb{R}^3$ look like geometrically.

1. Consider the following four points in the plane. This problem will demonstrate a couple ways that we could find a "line of best fit" for these four points. Part (a) is the usual method. The purpose of this exercise is to see how choosing a different "objective" (function to be minimized) can product different "lines of best fit" (essentially because it depends on what "best" means, which may be different in different applications).

$$(x_1, y_1) = (1, 1) \qquad (x_2, y_2) = (3, 2) \qquad (x_3, y_3) = (1, 6) \qquad (x_4, y_4) = (3, 7)$$

   (a) Suppose that we wish to find the coefficients c_1, c_2 that minimize the sum $\displaystyle\sum_{i=1}^{4}(c_1\, x_i + c_2 - y_i)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|$. Then find the optimal coefficients c_1, c_2. Sketch the four points and the line $y = c_1x + c_2$.

   (b) Suppose that we now want coefficients c_1, c_2 that minimize $\displaystyle\sum_{i=1}^{4}(c_1y_i + c_2 - x_i)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|$. Then find the optimal coefficients c_1, c_2 and sketch the four points and the line $x = c_1y + c_2$.

   (c) A third way to specify a line is using an equation of the form $c_1x + c_2y = 1$. Suppose that now we wish to find c_1, c_2 minimizing $\displaystyle\sum_{i=1}^{4}(c_1x_i + c_2y_i - 1)^2$. Identify vectors $\vec{v}_1, \vec{v}_2$, and $\vec{b}$ such that this is the same as minimizing $\|c_1\vec{v}_1 + c_2\vec{v}_2 - \vec{b}\|$. Then find the optimal coefficients c_1, c_2 and sketch the four points with the line $c_1x + c_2y = 1$.

   Comment: There is another objective function that is often used in practice to find lines (or more generally, linear spaces) of best fit: one can write the line as $c_1x+c_2y = c_3$, and attempt to minimize the sum $\sum_{i=1}^{4}(c_1x + c_2y - c_3)^2$. As stated, this problem has a silly solution: choose $c_1 = c_2 = c_3 = 0$. One can fix this by requiring that $c_1^2 + c_2^2 + c_3^2 = 1$. The solution to this problem is related to a topic called *singular value decomposition*, an extremely versatile idea in machine learning and data science. One benefit of this objective function is that it does not arbitrarily prefer horizontal error to vertical error, but rather treats the x and y axis on equal terms.

2. Prove that $\{\sin x, \cos x\}$ is a linearly independent set in $\mathcal{C}[-\pi, \pi]$. (*Hint:* There are many ways to approach this; be creative! One suggestion: if two functions are equal, then you can

evaluate them at any number to get two equal numbers. You can use this to get a large supply of equations of numbers from one equation of functions.)

3. (Textbook §3.1, problems 14-19)
   You can check answers to the odd-numbered problems in the back of the book. Problem 18 refers to the word "idempotent"; this means that $A^2 = A$.

4. (Textbook §3.1, problem 30)

5. (Textbook §3.2, problems 7-10)
   You can check the odd-numbered ones in the back of the book.

6. Suppose that A is an $n \times n$ matrix. Let $W \subseteq M_{n\times n}$ denote the set of matrices B such that $AB = BA$ (that is, A and B commute).
   (a) Show that W is a *subspace* of $M_{2\times 2}$.
   (b) Suppose that A is not a scalar multiple of the identity matrix. Find a set of two *linearly independent* elements of W.