INDUCTION

Discussion

Axiom 1 (Principle of induction)**.** *Let $\mathcal{S} \subseteq \mathbb{N}$ have the following properties:*

(1) $1 \in \mathcal{S}$
(2) *For all $s \in \mathcal{S}$, $s + 1 \in \mathcal{S}$.*

Then $\mathcal{S} = \mathbb{N}$.

We use this axiom as follows: Suppose you want to show that a statement $P(n)$ is true for all $n \in \mathbb{N}$. (This notation means that you have a statement into which you can feed any natural number n, to get a statement $P(n)$ about n. You would like to verify that $P(n)$ is a true statement for each $n \in \mathbb{N}$.) To do this, you have to first prove that $P(1)$ is true (this is called the *base case*). Then you assume that $P(n)$ is true, and try to use that fact to prove that $P(n+1)$ is true (the *inductive step*).

What does this accomplish? The first step tells you that $1 \in \mathcal{S}$, where $\mathcal{S}$ is the set of natural numbers n such that $P(n)$ is true. The second step tells you that if $n \in \mathcal{S}$, then $n + 1 \in \mathcal{S}$. The axiom tells you that $\mathcal{S} = \mathbb{N}$, i.e. that $P(n)$ is true for all $n \in \mathbb{N}$.

Example. Suppose we would like to show that for all $n \geq 1$:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

Base case: When $n = 1$, we have $2n - 1 = 1$, so the left side is 1, and the right side is $1^2 = 1$. So far, so good.

Inductive step: Suppose that for some $n \geq 1$, the claim is true. Let $S_n = 1 + 3 + \ldots + (2n - 1)$. We are interested in showing $S_{n+1} = (n + 1)^2$, assuming that $S_n = n^2$. Now, $S_{n+1} = S_n + (2n + 1)$. We have assumed that $S_n = n^2$, so combining these facts gives $S_{n+1} = n^2 + 2n + 1 = (n + 1)^2$, completing the inductive step and thus the proof.

Exercises

(1) Prove that, for all $n \in \mathbb{N}$,
$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$
(2) Prove that for all $n \in \mathbb{N}$, the number $4^n + 15n - 1$ is divisible by 9.
(3) Using the triangle inequality, prove that if $a_1, a_2, \ldots, a_n$ are real numbers, then
$$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|.$$
(4) Prove that for all $n \in \mathbb{N}$, the integer $6^n - 1$ is divisible by 5.
(5) Define a sequence (a_n) as follows: Let $a_1 = 1$ and let $a_{n+1} = \sqrt{a_n + 4}$ for $n \geq 1$. Prove that (a_n) is an increasing, bounded sequence. Deduce that (a_n) converges, and find its limit.