Math 281: Combinatorics • (Folsom)
Amherst College

Two Proofs of a First Theorem

Theorem. *For any $n \in \mathbb{N}_0$, we have that*

$$\sum_{k=0}^{n} P(n,k) = n! \sum_{k=0}^{n} \frac{1}{k!}.$$

Proof 1, Direct Proof. We have that

$$\sum_{k=0}^{n} P(n,k) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} = n! \sum_{k=0}^{n} \frac{1}{(n-k)!} = n! \sum_{j=0}^{n} \frac{1}{j!}.$$

Here, we have used that

- $P(n,k) = n!/(n-k)!$ for any n, k satisfying $0 \leq k \leq n$ and $n \in \mathbb{N}_0$,
- and we reindexed the last sum using $j = n - k$ (and since $0 \leq k \leq n$, we have that $0 \leq j \leq n$).

$\square$

Proof 2, Proof by Induction.

<u>**Base Case**</u> ($n = 0$): The LHS is:

$$\sum_{k=0}^{0} P(0,k) = P(0,0) = 1.$$

The RHS is:

$$0! \sum_{k=0}^{0} \frac{1}{k!} = 0! \cdot \frac{1}{0!} = 1 \cdot 1 = 1.$$

Hence, LHS=RHS for $k = 0$.

Inductive Hypothesis: Assume the theorem holds for some fixed $n \geq 0$. That is,

$$\sum_{k=0}^{n} P(n,k) = n! \sum_{k=0}^{n} \frac{1}{k!}.$$

Now, we must show it holds for $n + 1$. That is, we must show:

Claim.

$$\sum_{k=0}^{n+1} P(n+1,k) = (n+1)! \sum_{k=0}^{n+1} \frac{1}{k!}.$$

To prove the claim, recall that

$$P(n+1,k) = (n+1)P(n,k-1),$$

for $1 \leq k \leq n+1$. **We extract the $k = 0$ term from the sum to start** (because we can't apply the identity we just wrote down for $k = 0$) and obtain

$$\begin{aligned}\sum_{k=0}^{n+1} P(n+1,k) &= P(n+1,0) + \sum_{k=1}^{n+1} P(n+1,k) = 1 + \sum_{k=1}^{n+1} P(n+1,k) \\ &= 1 + \sum_{k=1}^{n+1}(n+1)P(n,k-1) = 1 + (n+1)\sum_{k=1}^{n+1} P(n,k-1) \\ &= 1 + (n+1)\sum_{j=0}^{n} P(n,j),\end{aligned}$$

where at the last step we have reindexed the sum, with $j = k-1$ (and since $1 \leq k \leq n+1$, we have that $0 \leq j \leq n$).

Now we apply the Inductive Hypothesis to the sum in blue, to obtain

$$1 + (n+1)\sum_{j=0}^{n} P(n,j) = 1 + (n+1) \cdot n! \sum_{j=0}^{n} \frac{1}{j!} = 1 + (n+1)! \sum_{j=0}^{n} \frac{1}{j!},$$

where we have also used that $(n+1) \cdot n! = (n+1)!$. Finally, we observe that

$$1 + (n+1)! \sum_{j=0}^{n} \frac{1}{j!} = (n+1)! \sum_{j=0}^{n+1} \frac{1}{j!},$$

proving the claim. (This is because the "1" appearing before the first sum in the line above can be realized as the $j = n+1$ term in the second sum in the line above (namely $(n+1)! \cdot \frac{1}{(n+1)!} = 1$).) □