Reading Section 2.1 of Harris-Hirst-Mossinghoff

1. (Textbook problem 2.1.1) In the C++ programming language, a variable name must start with a letter or the underscore character (_), and succeeding characters must be letters, digits, or the underscore character. Uppercase and lowercase letters are considered to be different characters.
   (a) How many variable names with exactly five characters can be formed in C++?
   (b) How many are there with at most five characters?
   (c) How many are there with at most five characters, if they must read exactly the same forwards and backwards? For example, kayak and T55T are admissible, but not Kayak.
2. (Textbook problem 2.1.2) Assume that a vowel is one of the five letters A, E, I, O, or U.
   (a) How many eleven-letter sequences from the alphabet contain exactly three vowels?
   (b) How many of these have at least one repeated letter?
3. (Textbook problem 2.1.5) A political science quiz has two parts. In the first part, you must present your opinion of the four most influential secretaries-general in the history of the United Nations in a ranked list. In the second part, you must name ten members of the United Nations Security Council in any order, including at least two permanent members of the council. If there have been eight secretaries-general in U.N. history and there are fifteen members of the U.N. Security Council, including the five permanent members, how many ways can you answer the quiz assuming you answer both parts completely?
4. (Textbook problem 2.1.8(a)-(e)) Compute the number of ways to deal each of the following five-card hands in poker.
   (a) Straight: the values of the cards form a sequence of consecutive integers. A jack has value 11, a queen 12, and a king 13. An ace may have a value of 1 or 14, so A 2 3 4 5 and 10 J Q K A are both straights, but K A 2 3 4 is not. Furthermore, the cards in a straight cannot all be of the same suit (a flush).
   (b) Flush: All five cards have the same suit (but not in addition a straight).
   (c) Straight flush: both a straight and a flush. Make sure that your counts for straights and flushes do not include the straight flushes.
   (d) Four of a kind.
   (e) Two distinct matching pairs (but not a full house).
5. (Textbook problem 2.1.9) According to the Laws of the Game of the International Football Association, a full football (soccer) team consists of eleven players, one of whom is the goalkeeper. The other ten players fall into one of three outfield positions: defender, midfielder, and striker. There is no restriction on the number of players at each of these positions as long as the total number of outfield players is ten.
   (a) How many different configurations are there for a full football team? For example, one team may field four strikers, three midfielders, and three defenders in addition to the goalkeeper. Another may play five strikers, no midfielders, and five defenders plus the goalkeeper.

(b) Repeat the previous problem if there must be at least two players at each outfield position.

(c) How many ways can a coach assign eleven different players to one of the four positions if there must be exactly one goalkeeper but there is no restriction on the number of players at each outfield position?

6. (Textbook problem 2.1.11) Suppose a positive integer N factors as $N = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$ where $p_1, p_2, \ldots, p_m$ are distinct prime numbers and $n_1, n_2, \ldots, n_m$ are all positive integers. How many different positive integers are divisors of N?

7. Use induction to prove for any $n \in \mathbb{N}$, that $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

8.

(a) How many ways can you rearrange the letters in

COCOMMMBIINBOBOMATAATTTORRRICRICRICOSOSOBOMBOMICS ?

(b) Generalize part a): how many ways can you rearrange the letters in a word that is m letters long, and contains n_j copies of the letter α_j, for each j satisfying $1 \leq j \leq k$ for some fixed k?

9. Prove for any $m \in \mathbb{N}$ and any $n \in \mathbb{N}_0$ that

$$\sum_{k=0}^{m-1} P(k, n) = \frac{P(m, n+1)}{n+1}.$$

Hint You may wish to use one of the following methods. Other methods are OK too. Pay attention to the different cases that may occur in any method, e.g., what happens if $n \geq m$?

(a) Prove using induction.

(b) Define the difference operator $\Delta(f(x)) := f(x+1) - f(x)$, and prove that for any $x \in \mathbb{N}$ and any $n \in \mathbb{N}_0$ that

$$\frac{\Delta(P(x, n+1))}{n+1} = P(x, n).$$

Use this result on Δ to establish the formula.

Note See also HHM §2.1 #14. The book's notation $x^{\underline{n}}$ means $P(x, n)$.