1. By a *standard n-sided die* we mean a die with n faces labeled $1, 2, 3, \cdots, n$. A non-standard n-sided die is a die with n faces labeled with any nonnegative[1] integers. If we have several dice (maybe of various sizes, maybe standard or nonstandard), the *generating function* of these dice is the sum $\sum_{k \geq 0} a_k x^k$, where a_k is the number of ways to roll a sum of k.

    (a) Find the generating function for a pair of standard 4-sided dice.

    (b) Find a nonstandard way to label two eight-sided dice, again with *positive integers* that gives the same outcomes with the same frequencies as a pair of standard eight-sided dice.

    (c) Show that it is possible to label the faces of three 4-sided dice with nonnegative integers in such a way that rolling all three and adding the numbers gives the same outcomes, with the same frequencies, as rolling two standard 8-sided dice. Give the labelings explicitly.

    (d) Describe a way you could label two 6-sided dice with nonnegative integers so that, if both are rolled and the numbers are added, the possible sums are 1 through 9 inclusive, with each occuring with the same probability. In this way you can use two six-sided dice to simulated a "standard nine-sided die."

2. Find a closed form for the generating functions for the following sequences:

    (a) $\{1, -1, 1, -1, 1, -1, \cdots, 1, -1\}$, where $1, -1$ is repeated a total of m times ($m \in \mathbb{N}$); i.e. the sequence has a total of $2m$ elements.

    (b) $\{a_n\}_{n=1}^{\infty}$, where $a_n := \frac{1}{n}$. That is, $\{a_1, a_2, a_3, \ldots\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

3. A tray of baked goods contains 4 donuts, 2 brownies, 1 cookie, 3 lemon squares, 1 cupcake, and 4 chocolate tortes. Suppose you want to select 6 baked goods from this tray - but you really like donuts, so at least two of those you select must be donuts. Moreover, you really dislike chocolate tortes, so none of those you select can be chocolate tortes.

    (a) Compute the number of ways to select 6 baked goods with these hypotheses "directly" as we have done in various similar in-class examples. i.e., Your answer should be expressed as a sum of products of binomial coefficients. Do not list all of the possibilities and then count them.

    (b) Determine the generating function for the sequence $\{b_n\}_{n=0}^{\infty}$, where $b_n :=$ # ways to select n baked goods with hypotheses as above. How could you instead find the answer to part (a) using this generating function?

4. Suppose a drawer contains ten red beads, eight blue beads, and eleven green beads. Determine a generating function that encodes the answer to each of the following problems.

    (a) The number of ways to select k beads from the drawer.

    (b) The number of ways to select k beads if one must obtain an even number of red beads, an odd number of blue beads, and a prime number of green beads.

    (c) The number of ways to select k beads if one must obtain exactly two red beads, at least five blue beads, and at most four green beads.

[1] in principle, there's no reason not to allow negative numbers or non-integer numbers, but we'd have to write our generating functions differently.

5. Use a combinatorial argument (rather than using generating functions) to count the number of different five-card hands that can be dealt from a triple deck, then the number of five-card hands that can be dealt from a quadruple deck.

6. (a) You have a deck of three cards marked by the following emojis: ☕, ☺, ☃. Call this your "emoji deck." Write down the generating function for the number of ways to select a hand of m cards from a double emoji deck.

   (b) Expand out this generating function from part a) by hand, so it is a single polynomial of the form $a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots a_rx^r$ for some $r \in \mathbb{N}_0$, and some coefficients $a_j \in \mathbb{N}_0$. For example, you will find that $a_2 = 6$.

   (c) Fill in the blank: "6 is the number of ways to ☐". Once you have filled in the blank, explicitly illustrate that 6 is the number of ways to by writing out the 6 ways to ☐ Your answers should have something to do with the emoji decks.

   (d) Repeat what you've done in part c) for every coefficient $a_j (1 \leq j \leq r)$ of your (expanded) polynomial generating function from part b).

7. (a) What is the generating function for the number of ways to select a hand of m cards from a triple deck, if there are n distinct cards in a single deck?

   (b) When $n = 30$ and $m = 7$, determine a combinatorial argument to find this same number 1 of ways to select a hand of $m = 7$ cards from a triple deck if there are $n = 30$ distinct cards in a single deck. This number of ways appears somewhere in one of the generating functions determined in part a). Where does it appear and in which generating function? (See also HHM §2.6.1 (p167) #4, which is similar.)