1. A *partition* of a positive integer n is a way to write n as a sum of one or more positive integers, where the order of the summands does not matter. For example, there are seven partitions of 4: 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. Let $p(n)$ denote the number of partitions of n. For example, the first several values of this function are $p(1)=1, p(2)=2, p(3)=3, p(4)=5$.

   Explain why the generating function for the sequence of integer partitions $\{p(n)\}_{n=0}^{\infty}$ has the following closed form:
$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{m=1}^{\infty} \frac{1}{1-x^m}.$$

   Hint. Use the geometric series: $\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \cdots$ (for $|z| < 1$). You may ignore all convergence issues in your answer to this problem. You should only explain (not fully "prove") why the series expansion for the the infinite product shown above has $p(n)$ as the coefficient of x^n for any $n \in \mathbb{N}_0$.

2. In each of the following problems, first compute the value of the expression for a few small values of n. Then use your data to conjecture a general formula. Last, prove that your formula is correct.

   (a) $\displaystyle\sum_{k=0}^{n} F_k$.

   (b) $\displaystyle\sum_{k=0}^{n} F_{2k}$.

   (c) $\displaystyle\sum_{k=1}^{n} F_{2k-1}$ if $n \geq 1$.

   (d) $F_{n+1}F_{n-1} - F_n^2$ if $n \geq 1$.

3. Suppose $a_0 = 0$, $a_1 = 5$, and $a_k = a_{k-1} + 6a_{k-2}$ for $k \geq 2$. Compute a closed form for the generating function of the sequence $\{a_k\}$. Then use this to determine a formula for a_k.

4. The Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$, and $L_k = L_{k-1} + L_{k-2}$ for $k \geq 2$. Find a formula for L_k in terms of ϕ and $\hat{\phi}$.

5. Solve the recurrence below for a_k using generating functions. Your answer should be an expression in terms of k.
$$a_k = 2a_{k-1} + 2^k \text{ for } k \geq 1, \text{ and } a_0 = 1.$$