1. Suppose that a sequence $\{a_n\}_{n\geq 0}$ has the following generating function:

$$G(x) = \frac{1-\sqrt[3]{1-3x}}{x}.$$

Show that

$$a_n = \frac{2\cdot 5\cdot 8\cdots(3n-1)}{(n+1)!} = \frac{1}{(n+1)!}\prod_{k=1}^{n}(3k-1).$$

2. Consider the following generalization of the Catalan numbers: let $C_{n,k}$ denote the number of sequences of $n+k$ symbols, consisting of n open parentheses (and k closing parentheses) such that, as the sequence is read from right to left, you have always seen at least as many (s as)s. In other words, these sequences are those that may be extended to a "nest of parentheses" as we discussed in class. Note in particular that $C_{n,n} = C_n$, the usual nth Catalan number.

   (a) Prove the following properties of the numbers $C_{n,k}$.
   - $C_{n,0} = 1$ for all $n \geq 0$.
   - If $n > k$ and $k > 0$, then $C_{n,k} = C_{n-1,k} + C_{n,k-1}$.
   - If $n = k$ and $k > 0$, then $C_{n,k} = C_{n,k-1}$.

   (b) Prove that for all n, k such that $n \geq k \geq 0$, the following formula holds:

   $$C_{n,k} = \binom{n+k}{k} - \binom{n+k}{k-1}.$$

   (Suggestion: proceed by induction on $n+k$.)

   (c) If A sequence of n opening parentheses and k closing parentheses is chosen at random, what is the probability that there are always at least as many (s as)s as you read from left to right? Simplify as much as possible.

   (d) Use part (b) to obtain a new proof of the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. This argument has the benefit of not requiring the use of generating functions.

> **Note** For the remaining problems, you may want to wait until we discuss the "cycle index" in class, or read about it in Section 2.7.3. It is not strictly necessary for these, but it simplifies the work involved in using Burnside's lemma.

3. Determine the number of different necklaces with 21 beads that can be made using four kinds of beads. Your equivalence classes should account for both rotations and flips.

4. Determine the number of different six-sided dice that can be manufactured using m different labels for the faces of the dice. Assume that each label may be used any number of times. (Suggestion: first, determine the cycle index for the group of symmetries of the faces of a cube.)