This document summarizes the content that will appear on the second midterm, as well as some suggestions as you review for it. I will also post a sample exam with a similar format and difficulty to the actual exam.

Exam format

- The exam will be in-class on Friday 11/30, so the time limit will be 50 minutes.
- You are allowed one page of notes, front and back, prepared any way you wish (typing, copying, sharing, etc. are all fine).
- The exam will most likely be five questions long.

Suggestions for your review

- Think hard about what to put on your note sheet, and how to organize it! This is a great way to study for the exam, since it will spur you to think holistically about the material, the most crucial ideas, and the places you've had the most trouble. It may well be that you won't even have to look at your sheet, because preparing it helped you thoroughy learn the material.
- Study your class notes in detail. Always *read actively.* Among other things, this means:
  1. Before reading a proof, try to summarize it on scratch paper. Even if you can't remember the details, this will help you prepare your brain to digest it.
  2. After each theorem, stop and ask yourself: what other theorems are similar or are proved in a similar way? Where do we apply it later in the course? What are some examples that we've done it class or on the homework?
  3. After each example, stop and ask yourself: what concepts of theorems is this illustrating? Can you generalize the example? What are similar examples elsewhere in the course?
- Review the homework assignments and posted solutions in detail, especially the problems you found challenging. When you review a problem, ask yourself: what concepts is this problem illustrating? What theorems from class does it apply or illustrate? What ideas do you need to come to mind in order to solve it?
- *After you've completing most of your review,* try the practice exam as a diagnostic and simulation of the timing. Remember that the practice exam, like the actual exam, will only cover essentially a random sample of the possible topics, so do not use it as a comprehensive review document!

Exam content

The list below provide a rough outline of the concepts we've studied in class or on the homework since the first midterm. I've provided references in many places to the spots where you can find more detail on the places where you need more review.

- Direct Productss of groups (§6, PSet 5)
  - The definition of $G \times H$.
  - How is $o((g,h))$ related to $o(g)$ and $o(h)$? How can you use this to tell whether or not $G \times H$ is cyclic? (cf. Theorem 6.1)
  - How can you express the Klein 4-group as (isomorphic to) a direct product? (Example from class; we didn't use the word "isomorphism" at the time, of course)
  - Under what conditions can you deduce that $G \times H$ is abelian? Non-ablelian? Cyclic? (cf. Theorem 6.1, Exercise 6.2,6.6, PSet 5 # 2,3)
- Cosets and equivalence relations (§9, PSet 5)
  - The definition of an equivalence relation, and the definition of cosets of G under a subgroup H.
  - The definition of the equivalence relation $\equiv_H$. What does $x \equiv_H y$ mean in terms of cosets? (cf. Theorem 9.3 and Corollary 9.4, PSet 5 # 9)
  - Is a *coset* ever also a *subgroup*?
  - Three useful equivalent statements: $Ha = Hb \Leftrightarrow a \in Hb \Leftrightarrow ab^{-1} \in H$. How do we write these statements differently when discussing cosets in a ring?
- Lagrange's theorem (§10, PSet 5)
  - What is the definition of the index $[G : H]$? How is it related to $|G|$ and $|H|$ (when these are finite)? (cf. p. 89 in Saracino)
  - The statement and proof of Lagrange's theorem. (Theorem 10.1)
  - Why does Lagrange's theorem quickly imply each of the following statements (in a finite group G)? (Theorems 10.4 and 10.6)
    - $g^{|G|} = e$ for all $g \in G$.
    - $o(g)$ divides $|G|$ for all $g \in G$.
    - If $|G|$ is prime, then G is cyclic.
- Quotient groups and normal subgroups (§11, PSet 6)
  - Define "normal subgroup."
  - Why are normal subgroups defined the way they are? Why would it be impossible to define a quotient group G/N without this definition ?(cf. discussion in Saracino beginning at the bottom of page 102)
  - Examples: what are the elements of $\mathbb{Z}/n\mathbb{Z}$? Of $Q_8/\{I, -I\}$? Which familiar groups are these isomorphic to?
  - Be comfortable doing computations in quotient groups (cf. PSet 6 #7 and 9; cf. also PSet 9 #7, where you do the same sort of thing in a quotient ring instead)

  - What is the order of G/N?
  - Examples of normal subgroups: $A_n \triangleleft S_n$, $SL(2,\mathbb{R}) \triangleleft GL(2,\mathbb{R})$, $Z(G) \triangleleft G$, any subgroup of an abelian group is normal.

- Group homomorphisms (§12, PSet 7)
  - Definitions: homomorphism, isomorphism.
  - Compositions of homomorphisms (resp. isomorphisms) are again homomorphisms (resp. isomorphisms). (cf. Theorem 12.1)
  - If $\phi : G \to K$ is a group homomorphism, what can you say about $\phi(g^{-1}), \phi(g^n), \phi(e_G)$ and $o(\phi(g))$? What can you say if you know that ϕ is an isomorphism? (cf. Theorems 12.4, 12.5)
  - There is a surjective group homomorphism from $\mathbb{Z}$ to any cyclic group. Why? (cf. example from class 10/23)
  - Any two cyclic groups of the same order are isomorphic. Why? (cf. Theorem 12.2)
  - Given a group homomorphism $\phi : G \to K$, and subgroups $H \le G, J \le K$, define the image $\phi(H)$ and inverse image $\phi^{-1}(J)$. These are subgroups; why? Under what hypotheses can you conclude that these are *normal* subgroups? (cf. Theorem 12.6)

- The fundamental theorem of group homomorphisms (§13, PSet 7)
  - Know the fundamental theorem, as stated in Saracino (Theorem 13.2). It is good to know the more general version stated in class (for non-surjective homomorphisms) especially if you are taking the Comps, but this will not be needed for this exam.
  - Define $\ker\phi$ (or a group homomorphism ϕ). Why is it a normal subgroup? (cf. Theorem 13.1)
  - We said (informally) in class that "$\ker\phi$ measures the failure of ϕ to be injective." What does this mean, more formally? What does $\ker\phi$ say about the cases where $\phi(x) = \phi(y)$?
  - Define the *induced homomorphism* $\overline{\phi}$ that we discussed in class. Why is it well-defined? Why is it a group homomorphism? Why is it injective? (cf. the proof of Theorem 13.2 in Saracino, especially the picture on page 123)
  - Examples: use the fundamental theorem to show that an order n cyclic group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$; the Klein 4-group is isomorphic to a quotient of Q_8 (by what normal subgroup?), the unit group of $\mathbb{R}$ is isomorphic to $GL(2,\mathbb{R})/SL(2,\mathbb{R})$ (we didn't call it the "unit group" at the time, of course; we wrote $(\mathbb{R} - \{0\}, \cdot)$ instead).

- Rings and field (§16, PSet 8)
  - Definitions: ring 0_R, 1_R, unit, unit group, field, integral domain, zero-divisor, nontrivial (ring, zero divisor).
  - Basic algebraic rules in rings for working with additive inverses, subtraction, and multiplication by integers (cf. Theorem 16.1, 16.4).
  - Specific comment: the book does not use the notation $R^\times$ for the unit group, but we use it frequently. Make sure you are familiar with it.
  - What are some examples of: integral domains, rings without unity, non-commutative rings, fields, rings with nontrivial zero-divisors?

- Fix an integer n. Which elements of $\mathbb{Z}_n$ are units? When is $\mathbb{Z}_n$ a field? (cf. Exercise 16.9, and discussion in class on 10/31 and 11/2).
- Under what circumstance does "cancellation" work for multiplication in rings? (cf. Theorem 16.5)
- Definition of direct sums $R \oplus S$ of rings.

• Subrings, ideals, and quotient rings (§17, PSet 9,10)

1. Know our definitions from class of rings and ideals (nonempty, closed under subtraction, and either closed under multiplication or sticky), and why they are equivalent to the definitions in the book.
2. What are some examples of subrings? Of ideals? Of subrings that are not ideals?
3. Why is R/I a well-defined ring when I is an ideal? (cf. our discussion from class, or the discussion in Saracino beginning in the second half of p. 167).
4. Be comfortable doing computations in quotient rings by working with coset representatives (cf. PSet 9 #6, 7, 8).
5. Definitions of prime and maximal ideals.
6. I is a prime ideal iff R/I has what property? (cf. Theorem 17.5)
7. If R is a commutative ring with unity: I is a maximal ideal if and only if R/I has what property? (cf. Theorem 17.7)
8. What are the prime ideals of $\mathbb{Z}$? What are the maximal ideals of $\mathbb{Z}$? (cf. Example 1 on p. 172, or our discussion in class).

• Ring homomorphisms (§18, PSet 10)

For the exam, you are only responsible for **the material up to the fundamental theorem (Theorem 18.5)** in §18. Any material covered on the exam will be covered either by Monday's class (11/26) or on Problem Set 10.

- Definitions of ring homomorphism and isomorphism.
- Basic algebraic homomorphisms: what can you say about $\phi(0_R)$, $\phi(na)$ (where $n \in \mathbb{Z}$) and $\phi(a^n)$ (where $n \geq 1$)?
- Criteria for $\phi(1_R) = 1_S$ (Theorem 8.2) and consequences (Theorem 18.1(iv)).
- Why is $\ker \phi$ an *ideal* for a ring homomorphism ϕ?
- Understand the statement and proof of the funamental theorem of ring homomorphisms (18.5).