MATH 350, Spring 2018

(With some revisions due to different coverage this semester)

Final Exam

1. (25 points) Determine whether each of the following statements are true or false. Circle your answer. You do not need to explain your answer, but an incorrect answer with some explanation may earn partial credit.

(a) $U(10)$ is cyclic, where $U(10)$ is the group of multiplicative units in $\mathbb{Z}_{10}$. (we called this $\mathbb{Z}_{10}^{\times}$).

(b) S_n/A_n is cyclic for all $n \in \mathbb{N}$.

(c) Let R be a ring. For all $a, b, c \in R$ with $a \neq 0$, if $ab = ac$ and then $b = c$.

(d) All ideals in $\mathbb{Z}$ are principal.

(e) $\mathbb{Z}_2[x]/(x^3+x+1)$ $\cancel{\mathbb{Q}[x]/\langle x^4 - 9x^2 + 12x + 15\rangle}$ is a field, where (x^3+x+1) $\cancel{\langle x^4 - 9x^2 + 12x + 15\rangle}$ is the principal ideal generated by $\cancel{x^4 - 9x^2 + 12x + 15}$ x^3+x+1.

2. (20 points)

(a) Consider the permutation $\sigma = (1\ 5\ 6)(1\ 3\ 4)^{-1}(2\ 5\ 4\ 6)(3\ 7)^{-1}$ in S_7.

i. Find the order of σ.

ii. Is $\sigma \in A_7$?

(b) Suppose $\tau \in A_7$ such that

$$\tau(1) = 2,\ \tau(2) = 4,\ \tau(3) = 7,\ \tau(6) = 6,\ \tau(7) = 3.$$

Then we must have $\tau(4) =$ _____ and $\tau(5) =$ _____. Explain your answer.

(c) Show S_n is not abelian for each $n \geq 3$.

3. (20 points) Prove each of the following.

(a) If G is a group of order p^2 where p is prime then every proper subgroup of G is cyclic.

(b) Let $\phi : \mathbb{Z}_{24} \to \mathbb{Z}_8$ be a group homomorphism. If $\ker \phi = \{0, 8, 16\}$ then ϕ is surjective.

(c) If K is the kernel of the group homomorphism $\phi : G \to H$ where $|G| = 6$ and $|H| = 10$ then the order of K is 3 or 6.

4. (20 points) Let G be a group and let H be a subgroup such that $x^{-1}y^{-1}xy \in H$ for all $x, y \in G$.

(a) Prove that H is a normal subgroup of G.

(b) Prove that G/H is abelian.

5. (20 points) Let $\phi : R \to S$ be a ring homomorphism. Let P be a prime ideal of S. Recall, $\phi^{-1}(P) = \{r \in R \mid \phi(r) \in P\}$.

(a) Prove that $\phi^{-1}(P)$ is an ideal of R.

(b) Prove that $\phi^{-1}(P)$ is a prime ideal of R.

6. (20 points) Consider $I = \langle x^2 + 4x + 2 \rangle \in \mathbb{Z}_7[x]$.

(a) Factor $x^2 + 4x + 2$ into irreducible polynomials in $\mathbb{Z}_7[x]$.

(b) Find a proper ideal J of $\mathbb{Z}_7[x]$ such that $I \subsetneq J$.

(c) Find a zero divisor in $\mathbb{Z}_7[x]/I$.

(d) Let $g(x) = x^3 + 2x - 1$ find a polynomial $r(x) \in \mathbb{Z}_7[x]$ with degree < 2 such that $g(x) + I = r(x) + I$ (Hint: think about what it means for $g(x) + I = r(x) + I$).

7. (25 points) Give an example of each of the following or explain why none exists.

(a) Two groups of order 5 which are not isomorphic.

(b) A commutative ring with unity that is not an integral domain.

(c) A prime ideal that is not maximal.

We covered the ideas needed to solve this, but didn't emphasize this type of problem. You may learn something from it, but it's safe to skip.

~~(d)~~ Two different factorizations for $x^2 + x$ in $\mathbb{Z}_6[x]$ (this means neither factor in one factorization is a unit multiple of either factor in the other).

(e) A root of $x^4 - 2$ in $\cancel{\mathbb{Q}}$ $\mathbb{Z}_5$.