Anything marked "suggestion" need not be handed in (but feel free to ask about it at help hours or by email!).

Read: Saracino, §0 − 2.

- **Suggestion:** Work (or think about) the following problems, especially if they concern a topic that is new or less comfortable. Problems marked with a * have answers provided at the back of the book.
  - §0: 7, 15
  - §1 : 3*, 6*, 9*
  - §2 : 1*, 5*
- **Suggestion:** When you read §0, stop before reading the proof of Theorem 0.1, and try to write out a proof yourself. Then read the book's proof and compare them.

  I suggest doing this often; it's a great habit every time you read a math text (I do it all the time today!). You won't always be able to find a proof, of course, but the effort will always "till the soil" in your mind so that you're ready to understand the book's proof.

Problems on Sets and Induction These problems concern material that we did not explicitly cover in class. Some of it may be new to you; if so, read §0 carefully, and do not be shy to come and ask for help at any of the help hours (see the course website for the up-to-date schedule).

1. Let A, B, C be sets. Prove that
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$
(This is sometimes phrased: "$\cup$ distributes over $\cap$.")

2. Suppose that x is a nonzero real number such that $x + \frac{1}{x}$ is an integer. Prove by induction that $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$.

3. Define the *Fibonacci sequence* f_1, f_2, f_3 as follows:
$$f_1 = f_2 = 1,\ f_3 = 2,\ f_4 = 3,\ f_5 = 5,\ f_6 = 8, \cdots$$
and in general
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3.$$
Prove that for all $n \geq 1$, $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$. (*Hint:* use induction.)

Problems on binary operations and groups

4. Suppose that $(G, *)$ is a group, and $x, y \in G$ are elements of G such that $x * y = x$. Prove that $y = e$, the identity element.

5. Give an example of a set S and binary operation $*$ such that $s * (s * s)$ isn't always equal to $(s * s) * s$. (*Suggestion:* you can either write down a formula, or write down an explicit table for a finite set, such as in exercises 1.9, 2.5, and 2.6).

6. Consider the following operation, on the set $S = (-1, 1)$:

$$a * b = \frac{a+b}{1+ab}.$$

   (a) Verify that $*$ is a binary operation on S.

   (b) Explain why $*$ is *not* a binary operation on $[-1, 1]$.

   (c) Verify that $(S, *)$ is a group.

   Note: This operation is sometimes called "relativistic addition." It tells how to add velocities, expressed as fractions of the speed of light, in special relativity. The fact that $a * b \in (-1, 1)$ means that, in special relativity, two velocities can do not "add" to more than the speed of light, even if they themselves are very close to it.

7. Let $*$ be an *associative* operation on a set S, and let $s_1, s_2, \cdots, s_n$ be elements of S. Prove that

$$s_1 * s_2 * \cdots * s_n$$

   has an unambiguous meaning, in the sense that no matter which way we insert parentheses into the expression, the result is the same.

   Hint: Prove, by induction on n, that any arrangement of parentheses gives a result that is equal to $s_1 * (s_2 * (s_3 * \cdots * (s_{n-1} * s_n) \cdots))$.

8. Denote by $\mathrm{SL}(2, \mathbb{R})$ the set of 2×2 matrices A with real entries such that $\det A = 1$. Prove that $(SL(2, \mathbb{R}), \cdot)$ is a group (here, $\cdot$ denotes matrix multiplication). (This is called the "special linear group." It is related to the group $\mathrm{GL}(2, \mathbb{R})$ that we considered in class, which is called the "general linear group.")

9. Suppose that $(G, *)$ is a group, and that $x * x = e$ for all $x \in G$ (here e is the identity element). Prove that $(G, *)$ is abelian (that is, that $*$ is commutative).

   Note: the original version of this problem set wrote x^2 rather than $x * x$. The two notations are synonymous; the notation x^2 will be more common later in the course (it is introduced in Chapter 4 of the book).

Important notes:

- Refer to the submission instructions on the Course Survey for how to submit your assignment on Gradescope.
- Please ask me for help if you find that it is taking more than a couple minutes to scan and submit your work.
- You are encouraged to work in groups while solving the problems, but all submitted work must be your own work in your own words.