1. [12 points] Let G be a group, and N a normal subgroup such that $[G : N] = m$. Prove that for all $g \in G$, $g^m \in N$.

2. [12 points] Let R denote the ring of all 2×2 matrices with real entries. Let S denote the following subset of R.

$$S = \left\{ \begin{pmatrix} a & -5b \\ b & a \end{pmatrix} : \ a, b \in \mathbb{Z} \right\}$$

   (a) Prove that S is a subring of R.

   (b) What element is the additive identity 0_S? Does S has a multiplicative identity 1_S?

3. [12 points] Let G be a finite group, and suppose that we have an action of G on a set Ω.

   (a) Suppose $\alpha \in \Omega$. Define the *stabilizer* $\mathrm{Stab}_G(\alpha)$ of α, and prove that it is a subgroup of G.

   (b) Suppose that $|G| = 27$ and $|\Omega| = 10$. Prove that there exists at least one element $\alpha \in \Omega$ such that $\mathrm{Stab}_G(\alpha) = G$.

   Hint: use the fundamental counting principle.

4. [12 points] Let R be a commutative ring with unity. Suppose that $a \in R$ satisfies $a^2 = a$ and $a \neq 0_R, 1_R$.

   (a) Prove that a is a zero-divisor.

   (b) Define $\langle a \rangle = \{ar : \ r \in R\}$ and $\langle 1_R - a \rangle = \{(1_R - a)r : \ r \in R\}$ as usual. Prove that the map $\phi : R \to \langle a \rangle$ given by

$$\phi(r) = ar$$

   is a ring homomorphism. Carefully identify any places in your argument where you use the assumption that R is commutative.

   (c) Prove that $R/\langle 1_R - a \rangle$ and $\langle a \rangle$ are isomorphic rings.

   Suggestion: first prove that $\ker \phi = \langle 1_R - a \rangle$.