F 9/27
class 15

Two possible defns of $\langle X \rangle$ (the subgroup generated by a set X).

(recall: informally, $\langle X \rangle$ = the smallest subgroup $H \supseteq X$; we write $\langle a_1, a_2, \cdots, a_n \rangle$ as shorthand for $\langle \{a_1, \cdots, a_n\} \rangle$.)

Fix group G & subset $X \subseteq G$.

① Conceptual: let

$$H = \bigcap \{ J \leq G \text{ st. } X \subseteq J \}. \text{ in other words,}$$

(NB this intersection includes at least one subgroup, namely G itself. So well-defined.)

$$= \{ g \in G : \forall \text{subgroup } J \leq G \text{ containing } X,\ g \in J \}.$$

Lemma 1) H is a subgroup. ~~containin~~

2) $H \supseteq X$ & $H \subseteq J$ for any other subgroup containing X.

omitted in class. instead commented that intersections of subgroups are subgroups (to be proved on HW).

Pf 1) $\forall a, b \in H$, $\forall J \leq G$ containing X,

$a, b \in J$ since $H \subseteq J$ (part of the intersection)

$\Rightarrow ab \in J$ (closure of J)

So since $ab \in J$ for all subgroups J containing X, $ab \in H$.

$\Rightarrow$ H is closed under mult.

Also, $\forall a \in H$, $\forall J \leq G$ cont. X, $a^{-1} \in J$ (closure of J)

so $a^{-1} \in H$ as well. $\Rightarrow$ H is closed under inverse.

Finally, $\forall J \leq G$ containing X, $e_G \in J$.

So $e_G \in H$ & therefore $H \neq \emptyset$.

So H is a subgroup.

obs edge case: $X = \emptyset$ means $H = \{e_G\}$. (smallest subgroup of all)

2) $\forall x \in X$, $x \in J$ for all $J \leq G$ containing X. So $x \in H$.
Therefore $X \subseteq H$.

$\forall J \leq G$ containing X, J is among the subgroups being intersected to form H, so $J \supseteq H$.

Item (2) justifies the use of the word "smallest."

② Constructive.

Let $Y = X \cup \{x^{-1} : x \in X\}$, & define

$$K = \{y_1 y_2 \cdots y_\ell : \ell \geq 0 \text{ \& each } y_i \in Y\}.$$

// interpret the "empty product" ($\ell=0$) to mean e_G.

K includes all elts. of G that closure forces to be present in a subgroup, once that subgroup includes X.

Lemma 2. K is a subgroup, containing X

Pf <u>K nonempty</u> since empty product ($\ell=0$) gives $e_G \in K$.

<u>K closed under mult.</u> since $\forall y_1, \ldots, y_\ell \in Y$, $\forall y_1', \ldots, y_{\ell'}' \in Y$,

$(y_1 y_2 \cdots y_\ell) \cdot (y_1' y_2' \cdots y_{\ell'}') \in K$. (product of $\ell + \ell'$ terms)

<u>K closed under inverse</u> since $\forall y_1, \ldots, y_\ell \in Y$,

$(y_1 \cdots y_\ell)^{-1} = y_\ell^{-1} y_{\ell-1}^{-1} \cdots y_1^{-1} \in K$ since each $y_i^{-1} \in Y$.

<u>K contains X</u> by the $\ell = 1$ case.

as we'd hope, then definitions agree:

Lemma 3 ~~$H=K$~~. $K \subseteq J$ for any subgroup $J \leq G$ containing X.

Pf ~~"$\subseteq$" follows since K is a subgroup of containing X (L.2) & Lemma 1(2) therefore gives $H \subseteq K$.~~

~~"$\supseteq$"~~ Suppose $y_1, \dots, y_\ell \in Y$. We claim $\cancel{g} = y_1 y_2 \cdots y_\ell \in \cancel{H}\ J$.

By induction on ℓ:

base case $\ell=0$: $e_G \in J$ since J is a subgroup.

inductive step suppose $\ell > 0$ & any product of $\ell - 1$ terms from Y is in J

Then

$$y_1 y_2 \cdots y_\ell = (y_1 y_2 \cdots y_{\ell-1}) \cdot y_\ell,$$

$y_1 y_2 \cdots y_{\ell-1} \in J$ by ind. hypothesis,

$y_\ell \in H$ since either $y_\ell \in X$ $(\Rightarrow y_\ell \in J)$

or $y_\ell^{-1} \in X$ $(\Rightarrow y_\ell = (y_\ell^{-1})^{-1} \in J$ by closure of J).

$\Rightarrow$ by closure of H under mult.,

$y_1 y_2 \cdots y_\ell \in H$.

Cor $K = H$

Pf $K \subseteq H$ by lemma 3,

& $H \subseteq K$ by lemma 1(2).

[Both K, H are subgroups containing X (Lemmas 1 & 2).

Defn This subgroup $K=H$ is denoted $\langle X \rangle$, & called the subgroup generated by X.

topological space : group
closed set : subgroup
limit of Cauchy seq. : product of ℓ terms

<u>Aside</u> (if you know analysis or point-set topology)

These two defn's are analogous to the two equivalent definitions of the <u>closure</u> of a subset $X \subseteq \mathbb{R}$:

① conceptual: $\overline{X} := \bigcap \{ F \subseteq \mathbb{R} : F \text{ closed } \& \ F \supseteq X \}$

② constructive: $\overline{X} = \{ \lim_{n\to\infty} y_n : y_n \in X \text{ for all } n, \ \& \ (y_n)_{n\geq 1} \text{ is a Cauchy sequence} \}$.

this sort of pair of equivalent definitions is very common in higher mathematics.

<u>eg</u> in $GL(2,\mathbb{R})$, let $H = \langle \overbrace{\begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}}^{X} \rangle$.

<u>claim</u> $H = \left\{ \begin{pmatrix}1&n\\0&(-1)^m\end{pmatrix} : m,n \in \mathbb{Z} \right\}$.

<u>pf</u> "$\subseteq$" because the RHS is a subgroup containing X.

<u>nonempty</u>: contains $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $(m=n=0)$

<u>closed under mult</u>: $\begin{pmatrix}1&n\\0&(-1)^m\end{pmatrix} \cdot \begin{pmatrix}1&n'\\0&(-1)^{m'}\end{pmatrix}$

$$= \begin{pmatrix}1 & n'+(-1)^{m'}\cdot n\\ 0 & (-1)^{m+m'}\end{pmatrix} \in \text{RHS}.$$

<u>closed under inverse</u>: $\begin{pmatrix}1&n\\0&(-1)^m\end{pmatrix}^{-1} = \begin{pmatrix}1 & (-1)^{m+1}\cdot n\\ 0 & (-1)^m\end{pmatrix}$.

"$\supseteq$" because $\forall m,n \in \mathbb{Z}$,

$$\begin{pmatrix}1&n\\0&(-1)^m\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}^m \cdot \begin{pmatrix}1&1\\0&1\end{pmatrix}^n \in H$$

by the <u>constructive</u> defn. of $\langle X \rangle$.

eg. in S_n,

a) let C = set of all cycles $(a_1, a_2, \cdots, a_\ell)$.

Then $\langle C \rangle = S_n$ since any $f \in S_n$ has a cycle decomp.

"cycles generate S_n".

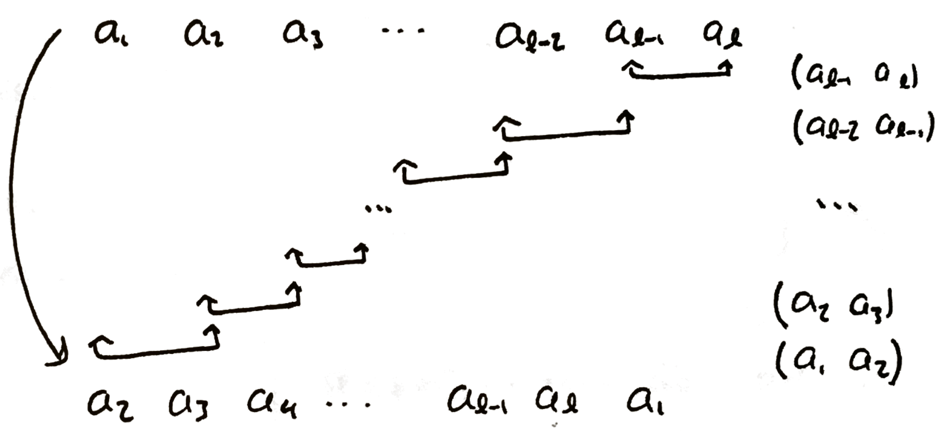
b) let T = ~~set of~~ transpositions (2-cycles) $\{(a,b) : a, b \in [n]\}$

Monday

observe. $\forall$ cycle $f = (a_1, \cdots, a_\ell)$

$$f = (a_1\ a_2)(a_2\ a_3)(a_3\ a_4) \cdots (a_{\ell-1}\ a_\ell)$$

visually:



$\Rightarrow f \in \langle T \rangle$ 4

$\Rightarrow C \subseteq \langle T \rangle$ (transpositions generate all cycles)

$\Rightarrow$ any subgroup containing T contains C,
hence contains all of S_n!

$\Rightarrow \langle T \rangle = S_n$ as well.

so S_n is also generated by transpositions.

//cf: the "bubblesort" algorithm

c) let $A = \{(a, a+1) : a \in \{1, 2, \cdots, n-1\}\}$.

"adjacent transpositions"

In fact, these also generate S_n. // put on PSet 5.

cf "bubblesort": way to re-order an array in ascending order
(not the most efficient by far for long arrays!)

given sequence $f(1), f(2), \cdots, f(n)$,

find an index a s.t. $f(a) > f(a+1)$

& swap these values (i.e. replace f by $f \circ (a, a+1)$).

continue until $f(1) < f(2) < \cdots < f(n)$.

Two other constructions of subgroups

1) centralizer of an element or set:

$$C_G(x) = \{a \in G : ax = xa\}.$$

it's a subgroup:

$e \in C_G(x) \Rightarrow$ nonempty.

if $a, b \in C_G(x)$, then $abx = axb = xab$

$\Rightarrow ab \in C_G(x)$. closed under mult.

if $a \in C_G(x)$, then $a^{-1}x = xa^{-1}$

$\Rightarrow aa^{-1}xa = axa^{-1}a$

$\Rightarrow xa = ax$

$\Rightarrow a^{-1} \in C_G(x)$. closed under inverse.

resume here Monday.

for a set X, define $C_G(X) = \bigcap_{x \in X} C_G(x)$.

(HW: check that intersection of subgroups is a subgroup).

in quantum mechanics: $x \in G$ is an observable, & $C_G(x)$ = all simultaneously measurable observables.

2) <u>center</u> of the group: $Z(G) = \{z \in G : \forall g \in G,\ gz = zg\}$.

This is equiv. to $C_G(G) \Rightarrow$ also a subgroup.

<u>eg</u>. in $\overset{G}{\overset{\shortparallel}{GL(2,\mathbb{R})}}$,

$$C_G\left(\begin{pmatrix}1&0\\0&2\end{pmatrix}\right) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \in GL(2,\mathbb{R}) : \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix} = \begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\right\}$$

$$= \left\{ \quad " \quad : \begin{pmatrix}a&2b\\c&2d\end{pmatrix} = \begin{pmatrix}a&b\\2c&2d\end{pmatrix}\right\}$$

$$= \left\{\begin{pmatrix}a&0\\0&d\end{pmatrix} : a, d \in \mathbb{R}\right\}.$$

more intrinsically: set of mats. w/ the <u>same eigenspaces</u>.

(important observation in quantum mechanics).

<u>eg</u> in $GL(n,\mathbb{R})$,

$Z(G) = \{c \cdot I : c \in \mathbb{R}^\times\}$.

(see if you can prove it!)