1. Let G be a finite group, and let g, h be two elements of G that *commute*, i.e. $gh = hg$.

   (a) Prove that if $\langle g\rangle \cap \langle h\rangle = \{e\}$, then $o(gh)$ is equal to the least common multiple of $o(g)$ and $o(h)$.

   Suggestion (not to hand in): use part (a) to solve problem 3.1.3 (prove that the order of a permutation is the least common multiple of its disjoint cycle lengths).

   (b) Prove that $\gcd(o(g), o(h)) = 1$, then $\langle g\rangle \cap \langle h\rangle = \{e\}$, and therefore that $o(gh)$ is the least common multiple of $o(g)$ and $o(h)$.

   Hint for (b): use Lagrange's theorem.

2. Let $G = \mathrm{GL}(n, \mathbb{R})$. This problem explores the link between centralizers in G and eigenvectors, generalizing our discussion in class of the centralizer of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

   (a) Let $G = \mathrm{GL}(n, \mathbb{R})$, and suppose that $A, B \in G$ are two matrices, with $B \in C_G(A)$. Suppose also that $\lambda \in \mathbb{R}$ is an eigenvalue of A with $\dim \ker(A - \lambda I) = 1$. Prove that if $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $\vec{v}$ is also an eigenvector of B (not necessarily with the same eigenvalue!).

   (b) Suppose that A has an eigenbasis $\{\vec{v}_1, \cdots, \vec{v}_n\}$ (that is: a basis of $\mathbb{R}^n$ consisting of eigenvectors of A), each element of which has a *different eigenvalue* (this happens if and only if the characteristic polynomial of A has no repeated roots). Prove that

   $$C_G(A) = \{B \in G : \quad \textit{each } \vec{v}_i \textit{ is also an eigenvector of } B\}.$$

   (This implies that for all $B \in C_G(A)$, the matrices A and B are *simultaneously diagonalizable.*)

   (c) Deduce from part (b) that if A is a diagonal matrix with all entries on the diagonal distinct, then $C_G(A)$ consists of all diagonal matrices (this directly generalizes the example from class).

3. (**4.4.6**) Give a one sentence proof that conjugacy classes of a group partition the group. Then find all the conjugacy classes of D_8.

4. (**4.4.10**) Let $G = \{a, b, c, d, e, f\}$, and let $\Omega = \{x, y, z, u, v, w\}$. We know that G is a group and Ω is a set. We also know that G acts on Ω. The following table tells us how every element of G acts on elements of Ω:

	x	y	z	u	v	w
a	w	y	z	v	u	x
b	x	u	z	v	y	w
c	w	v	z	u	y	x
d	x	y	z	u	v	w
e	x	v	z	y	u	w
f	w	u	z	y	v	x

So for example, $c \cdot x = w$, and $c \cdot u = u$.

(a) What is $(bc) \cdot y$?

(b) Can you find a subgroup of G with one element? What about a subgroup of G with two elements? What about a subgroup of G with three elements? If the answer is yes, give the elements of the subgroup, and in any case give adequate explanation for your answers.

(c) Can you find an orbit with three elements?

(d) Let H be the stabilizer of w in G. If we multiply $c \in G$ by every element of H—that is, find ch for all $h \in H$—we get what is called a *left coset* of H and denoted by cH. What are the elements of the left coset cH?

5. (**5.2.1**) If H and K are subgroups of G of order 75 and 242 respectively, what can you say about $H \cap K$?

6. (**5.2.4**)

   (a) Let G be a non-cyclic group of order 121. How many subgroups does G have? Why?

   (b) Can you generalize your result of the previous part?

7. (**5.2.3**) Suppose that a finite group G has an element g with order 7 and an element h with order 11. What is the minimum value of $|G|$?

8. I mentioned in class that an action of a group G on a set Ω is equivalent to a homomorphism $G \to \text{Perm}(\Omega)$. The purpose of this problem is to make that observation more precise.

   (a) Suppose that $\phi : G \to \text{Perm}(\Omega)$ is a group homomorphism. Prove that the map $G \times \Omega \to \Omega$ defined by the formula
   $$g \cdot x = \phi(g)(x)$$
   is a group action.

   (b) Conversely, suppose that we have a group action of G on Ω. For all $g \in G$, define a map $\phi(g) : \Omega \to \Omega$ by $\phi(g)(x) = g \cdot x$. Prove that $\phi(g)$ is a permutation of Ω, hence ϕ is a map $G \to \text{Perm}(\Omega)$. Prove that ϕ is a group homomorphism.

Some other good problems to try for additional practice (but not to hand in):
3.1.3, 4.3.8, 4.4.13, 4.4.16, 5.2.2