Below are all of the additional practice problems suggested on problem sets so far. These are all useful for exam review. I will put at least one of these problems (perhaps with minor modifications) on the exam.

1. (**1.1.3**) List the symmetries of an isosceles triangle.

2. (**1.1.4**)
   (a) List the symmetries of a rectangle.
   (b) Write the multiplication table for the symmetries of a rectangle.

3. (**1.2.2**) Let $\Omega = \mathbb{Z}$ be the set of integers. Define $f : \Omega \to \Omega$ by
$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd} \end{cases}.$$
Is $f \in \mathrm{Perm}(\Omega)$? If so, what is its inverse? What is f^2? What about f^3?

4. (**1.2.5**) Construct a complete multiplication table for S_3. What is the center (see Definition 1.7) of S_3? If $f = (1\ 2\ 3)$, what is $\mathbf{C}_{S_3}(f)$, the centralizer of f in S_3?

5. (**1.3.1**)
   (a) Find $-\frac{3}{4} - 4$ in $\mathbb{Z}/7\mathbb{Z}$.
   (b) In $\mathbb{Z}/12\mathbb{Z}$ does every non-zero element have a multiplicative inverse (i.e., for $a \in \mathbb{Z}/12\mathbb{Z}$ can we find b such that $ab = 1$)?
   (c) In $\mathbb{Z}/7\mathbb{Z}$ does every non-zero element have a multiplicative inverse?
   (d) We want to know for which integers $n > 1$ every non-zero element of $\mathbb{Z}/n\mathbb{Z}$ has a multiplicative inverse. Look at some examples and make a conjecture. You do not have to prove your conjecture.

   Comment: The textbook sometimes writes (as above) "$\frac{a}{b}$ in $\mathbb{Z}/n\mathbb{Z}$" as a shorthand for ab^{-1}. I usually avoid this notation since it has the potential to cause confusion.

6. (**1.3.2**) Consider the addition operation on $\mathbb{Z}/7\mathbb{Z}$. Start with the element $a = 3$ and find $2a = a+a$, $3a = a+a+a$, and so on until at least $20a$. Do you notice a pattern? Now change a to 4 and repeat what you did. Make a general conjecture based on the patterns that you found. Repeat what you did for $\mathbb{Z}/6\mathbb{Z}$. Is there any difference?

7. (**1.4.4**) How many elements does $\mathrm{GL}(2,3)$ have? Justify your answer without an appeal to Theorem 1.64. Can you extend your argument to $\mathrm{GL}(2,p)$ where p is an arbitrary prime?

8. (**1.4.6**) List the elements of $\mathrm{SL}(2,2)$? What are the possible values for a determinant of a matrix over $\mathbb{Z}/2\mathbb{Z}$? What can you say about the relationship between $\mathrm{GL}(n,2)$ and $\mathrm{SL}(n,2)$?

9. (**2.1.1**) Let I_n be the $n \times n$ identity matrix. Is
$$\{rI_n \mid r > 0, r \in \mathbb{R}\}$$
a group under matrix multiplication?

10. (**2.1.3**) Let $\mathbb{Z}$ denote the set of integers, and let
$$G = \{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a \in \mathbb{Z}\}.$$
Prove that G together with the usual matrix multiplication forms a group.

11. (**2.1.6**) Let n be a positive integer. For which n is S_n abelian? Prove your assertion.

12. (**2.2.3**) If G is a group in which $a^2 = e$ for all $a \in G$, show that G is abelian.

13. (**2.3.4**) Find the order of each of the elements of the group $((\mathbb{Z}/8\mathbb{Z})^\times, \cdot)$. Is this group cyclic? Do the same for the group $((\mathbb{Z}/10\mathbb{Z})^\times, \cdot)$.

14. (**2.3.9**) Let ℓ be an integer greater than 1, and let G be a finite group with no element of order ℓ. Can there exist $a \in G$ with $\ell \mid o(a)$? Prove your assertion.

15. (***2.3.16***) Let G be a group and let $x, y \in G$. Assume that $xy = yx$, $o(x) = p$, and $o(y) = q$, where p and q are distinct prime numbers. What can you say about $o(xy)$?

16. (**2.3.21**) Consider a fixed shuffle of a deck of cards. Does the repeating of this fixed shuffle some finite (positive) number of times bring the deck eventually back to its original order? Why?

17. (**2.4.3**) Are the groups $(\mathbb{Z}/12\mathbb{Z}, +)$ and $(\mathbb{Z}/13\mathbb{Z})^\times$ isomorphic?

18. (***2.5.7***)
    (a) Let m and n be integers greater than 1. What is the order of the element $(1, 1)$ in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$? Make a conjecture.
    (b) Under what conditions would $(1, 1)$ be a generator for $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$?

19. (**2.4.16**) Give an example of two groups G and H, an element $x \in G$, and a homomorphism $\phi : G \to H$ such that $o(x)$ does not equal $o(\phi(x))$.

20. (**2.5.11**) Assume that $G \times H$ is an abelian group. Can we conclude that G and H are abelian?

21. (**2.6.2**) Let $G = (\mathbb{Z}/12\mathbb{Z}, +)$. Find all subgroups of G.

22. (**2.6.3**) Find all subgroups of $(\mathbb{Z}/18\mathbb{Z}, +)$.

23. (**2.6.9**) Let G be a group, and assume that a and b are two elements of order 2 in G. If $ab = ba$, then what can you say about $\langle a, b\rangle$?

24. (**3.1.1**) Let $\sigma = (a_1\ a_2\ \cdots\ a_m) \in S_n$. Find σ^{-1}.

25. (**3.1.4**) What is the smallest positive integer n for which S_n has an element of order 15? What about an element of order 11?

26. (**3.1.5**) Does S_7 have a subgroup isomorphic to $\mathbb{Z}/12\mathbb{Z}$? Either prove that it does not, or exhibit such a subgroup.

27. (**3.2.2**) Let x and y be two three-cycles. Can xy be a four-cycle? Either give an example, or prove that it is impossible.

28. (**3.2.3**) Define $\phi : S_n \to \mathbb{Z}/2\mathbb{Z}$ by

$$\phi(x) = \begin{cases} 0 & \text{if } x \text{ is an even permutation,} \\ 1 & \text{if } x \text{ is an odd permutation.} \end{cases}.$$

    Show that ϕ is a group homomorphism.

29. (**3.2.5**) The alternating group A_6 has how many elements of order 3?

30. (**3.2.8**) Is A_4 isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z}$? Why?