Reading §1.1 and 1.2 of Leary-Kristiansen

The first part of this assignment is about **formal languages** and **formal (deductive) systems** as we have defined them in class. The textbook does not cover this topic exactly; it dives straight into what are called *first-order* languages. To answer the questions here, you should read carefully the notes from class this week and follow the definitions in those notes are precisely as you can. For the second part, you should read §1.2 in the textbook.

You should submit solutions to these problems on Gradescope. You may handwrite or type answers, but in any case they should be clearly and fully explained. Unless instructed otherwise, you should include proofs or explanations for any statements you make. Of course it's often hard to know how much detail to give, so please ask if you are unsure.

1. In the formal system ADDITION introduced in class, write down an explicit proof of the theorem
$$\heartsuit\heartsuit\heartsuit + \heartsuit\heartsuit = \heartsuit\heartsuit\heartsuit\heartsuit\heartsuit.$$

2. (a) Describe a formal system MULTIPLICATION, based on the formal language HEARTS introduced in class and with exactly one axiom, in which the theorems are precisely those formulas of the form
$$x + y = z$$
where the number of $\heartsuit$ symbols in the expression z is equal to the **product** of the number of $\heartsuit$ symbols in the expression x and the number of $\heartsuit$ symbols in the expression y. (You don't need to prove that your system gives the right theorems; just describe its axioms and rules of inference.)

   (You might be put off by the fact that we are still using the symbol + in this language, not some other symbol that reminds you more of multiplication. Don't be! There is no rule that the symbol + has to represent addition. In fact, in this problem, none of the symbols represent anything! They are just symbols that are manipulated according to the rules of the system.)

   (b) In your formal system from the part (a), write down an explicit proof of the theorem
$$\heartsuit\heartsuit\heartsuit + \heartsuit\heartsuit = \heartsuit\heartsuit\heartsuit\heartsuit\heartsuit\heartsuit.$$

   (c) Describe another formal system based on the formal language HEARTS, but with **no** rules of inference, whose theorems are exactly the same as those of the system MULTIPLICATION. (This is a bit of a trick question.)

3. Let L be the formal language in which the only symbol is $\bullet$ and where any word is a formula.

   (a) Consider the formal system E based on L in which there is one axiom: $\bullet\bullet$, and there is one rule of inference:
$$\text{premise: } x, \quad \text{conclusion : } x\bullet\bullet$$
   for any formula x.

   Give a simple rule for deciding if a formula in L is a theorem of E or not. Prove that your answer is correct.

(b) Consider the formal system D based on L in which there is one axiom: $\bullet$, and two rules of inference

$$(R1)\text{ premise: } x, \quad \text{conclusion: } x\bullet\bullet$$
$$(R2)\text{ premise: } x, \quad \text{conclusion: } x\bullet\bullet\bullet$$

for any formula x.

Give a simple rule for deciding if a formula in L is a theorem of D or not. Prove that your answer is correct.

4. Choose a set of mathematical objects you are familiar with, and write down the symbols of a first-order language that might be useful for working with those objects. You have to decide what constant, function and relation symbols would be appropriate, and you should explain what the intended meaning of those symbols would be. (You may not repeat an example from class or the textbook.)

   For example, if you chose sets as your objects (which you can't because I'm using them as an example here), then you might have

   - one constant symbol: $\varnothing$ (intended to be the empty set);
   - two binary function symbols: $\cap$ and $\cup$ (intended to be intersection and union);
   - two binary relation symbols: $\in$ and $\subseteq$ (intended to be the membership and subset relations).

   You might notice that in the textbook (Example 1.2.3), a first-order language for talking about sets is described that only has the binary relation symbol $\in$ and none of the others mentioned above. That illustrates that there can be different languages we could use for the same situation. The drawback of having fewer symbols is that it becomes more complicated to express even simple ideas. For example, compare how you might express the statement "the empty set is a subset of any set" in the language that only includes $\in$ versus a language which also has $\varnothing$ and $\subseteq$.